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Random Consumer Demand

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I present a theory of random consumer demand. The primitive is a collection of probability distributions on budgets. Axioms constrain these distributions, including analogues of preference axioms, such as transitivity, monotonicity and convexity. Results establish a complete representation of theoretically consistent demand. The theory's purpose is empirical application. To this end, the theory has desirable properties. Intrinsically stochastic, econometricians can apply it without adding extrinsic randomness in the form of errors. Random demand is parsimoniously represented by a single function on the consumption set. Finally, there exist practical methods for inference based on the theory, described in a companion paper.

Introduction

I begin with a familiar consumer demand environment. There are n consumer goods and a consumption set X which is the non-negative orthant $\mathbb{R}^n_+$. Vectors $x = (x_1, \ldots, x_n) \in X$ are bundles of these goods.

I add a consumer with income m facing a vector $w = (w_1, \ldots, w_n)$ of prices. The consumer can afford any bundle in the budget

$$B(w, m) \equiv \{x \in X : w'x \leqslant m\}.$$

I assume that the consumer has preferences that are complete, transitive, continuous, strictly monotone and strictly convex.

Figure 1 gives an example with $n = 3$ goods. It shows a budget simplex $B(w, m)$, some indifference curves on the budget frontier $\bar{B}(w, m) \equiv \{x \in X : w'x = m\}$ and the most preferred feasible demand, denoted (x_1^*, x_2^*, x_3^*).

In a commonly used approach to empirical consumer demand analysis, the practitioner chooses a class of preferences, and then finds the preference in this class for which optimal choices on observed budgets are closest to observed choices on these budgets. For example, the practitioner may choose, as a class of preferences, those preferences that rationalize demand systems with the AIDS (Almost Ideal Demand System) parametric form, introduced by Deaton and Muellbauer (1980). In this case a convenient and widely used measure of the closeness between optimal and observed demand is the sum of squared differences of expenditure shares. Finding the preference of best fit amounts to using ordinary least squares (OLS) to estimate the AIDS parameters.

In terms of our example, the closeness of an observed choice (x_1, x_2, x_3) to the optimal bundle (x_1^*, x_2^*, x_3^*) is measured by the Euclidean distance between $(x_1w_1/m, x_2w_2/m, x_3w_3/m)$ and $(x_1^*w_1/m, x_2^*w_2/m, x_3^*w_3/m)$.

I point out that this measure of closeness is unrelated to preference: the consumer may be far from indifferent between two bundles equally close to (x_1^*, x_2^*, x_3^*). Consider two deviations from (x_1^*, x_2^*, x_3^*) obtained by trading off one unit of the third good. If we exchange it for the first good, we move in the direction of A_1; for the second good, in the direction of A_2. In terms of Euclidean distance between expenditure shares, both

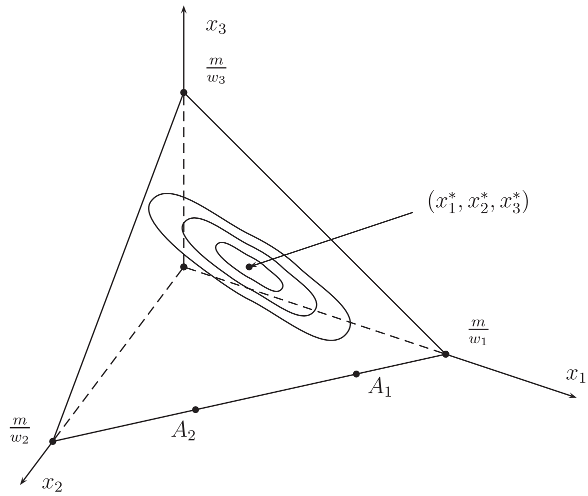

FIGURE 1. A consumer demand example

deviations are equally close to (x_1^*, x_2^*, x_3^*). However, the shape of the indifference curves tells us that the consumer is not indifferent between the two deviations: the area of the set of budget frontier bundles intermediate in preference between the two deviations is quite large.

Many would regard choice behaviour governed by regular preferences as ideal behaviour, close enough to real behaviour to be a useful model, but not the literal truth. Since ideal behaviour involves finding a choice to which no feasible alternative is preferred, it makes sense to measure how close real behaviour is according to how few feasible alternatives are preferred to the realized choice. This suggests that we compare the closeness of two non-optimal bundles in terms of their relative desirability, rather than their relative distances to the optimal choice. Varian (1990) argues for the former and gives some examples of goodness-of-fit measures.

In a more general context, we can adopt a goodness-of-fit interpretation for the likelihood-based statistical approaches that applied economists often use to estimate preference and technology parameters. These approaches require the specification of probability distributions for observables. Usually, however, economists view choice as deterministic. And so the practitioner often adopts a *measurement error* approach, in which probability distributions are said to govern errors between model predictions and observed choices. An implicit measure of the goodness-of-fit of observed choices to model predictions is the likelihood, the probability density of observable choices evaluated at their realized values.

In the measurement error approach, the implied goodness-of-fit is measured in the space of choices. But this is not an essential feature of likelihood-based statistical approaches. Instead, we can specify probability distributions for observable choices whose implied measure of fit is in terms of relative desirability. We just need to choose densities over feasible choices whose level curves are the agent's indifference curves. So for example, the indifference curves of Figure 1 would also be the contours of the density of observed choices. This approach has the benefit of measuring fit in terms of desirability, and the additional advantage of parsimony, since it simplifies the joint specification of preferences and densities.

This paper develops this idea further, and the result is a new theory of random consumer demand built on an axiomatic foundation. There is a utility-like representation for theoretically consistent demand, and it is a function on the consumption set giving choice distributions on all budgets.

The new theory is intimately related to classical consumer theory. Most assumptions are analogous to classical axioms such as transitivity, monotonicity and convexity. The theory is not exactly a generalization of the classical theory, but we have two theorems showing that it is a near generalization. Theorem C1 states that, for any classical demand behaviour rationalized by a complete, transitive, strictly monotone and convex preference $\succeq$ on the consumption set X, there is observationally equivalent random demand behaviour satisfying all but one of our axioms. Theorem C2 shows us that, even with the missing axiom imposed, we obtain a sort of limiting observational equivalence.

I begin with a primitive that is a collection of choice distributions, not a preference relation. I propose axioms that constrain these choice distributions. I show that theoretically consistent random demand has a representation in the form of a function on the consumption set. I call the representation an L-utility function, to invite intuitive comparison with utility, but to avoid confusion.

The purpose of this new theoretical approach to consumer demand is to improve work on empirical consumer demand problems. To this end, the theory has several advantages. First, the 'fit' of an observed choice is measured by the relative desirability of the choice and its feasible alternatives, rather than by some metric on the consumption set.

The representation is parsimonious. A single L-utility function on the consumption set not only describes how demand responds to changes in prices and income, as a utility function does, but also gives choice distributions on all budgets.

The representation facilitates inference. Our representation theorems identify theoretically consistent random demand with an L-utility function. The econometrician can therefore work directly with the L-utility function. I have developed a practical method for Bayesian non-parametric inference based on the theory, described in McCausland (2004).

Demand is intrinsically stochastic, and so the econometrician can apply the theory directly without recourse to error terms or random preferences. In usual practice, distributions of errors and preferences are given without theoretical justification.

Unlike standard consumer theory, the proposed theory does not rule out the violations of the axioms of revealed preference that are sometimes observed. The new theory is more forgiving, without being undisciplined.

I. Primitives

Since the primitive concept in my theory is somewhat non-standard in economics, I must digress briefly to situate the theory in the broader literature on theories of choice. I remind readers of two important distinctions. The first is the difference between theories of deterministic choice and theories of stochastic choice; the second is the difference between random preference theories and random choice theories.

In theories of deterministic choice, the primitive is usually a binary preference, either over a universe of objects or a set of probability distributions over a universe of objects. The former is typically used to analyse choice under certainty, and the latter to study choice under uncertainty. In both cases, however, choice itself is deterministic.

In stochastic theories the primitive is usually either a probability distribution over binary preferences or a collection of probability distributions, one for each possible choice set. Theories featuring the former are called random preference (or random utility) theories, while theories with the latter are called random choice theories.

In economics, theories of choice are usually deterministic. Stochastic theories are less common, but prevail in the literature on discrete choice. Almost all theories of stochastic choice in economics are random preference, rather than random choice, theories. Some exceptions include Debreu (1958), Georgescu-Roegen (1958), Halldin (1974), Bandyopadhyay *et al.* (1999, 2002, 2004). Georgescu-Roegen (1958) and Halldin (1974) are closest to the present paper, as they both concern the implications of stochastic versions of binary preference axioms on consumer demand distributions. The papers by Bandyopadhyay *et al.* take a revealed (stochastic) preference approach. All of these papers are theoretical, and none attempts to provide a theory ready for application to applied consumer demand problems.

In mathematical psychology, however, theories of choice are usually stochastic rather than deterministic. An important reason is that individuals in experimental situations do not always behave invariably, even in well controlled binary choice situations. Again in contrast to the economics literature, theories of stochastic choice in mathematical psychology are more often random choice theories than random preference theories. For further reading on theories of choice in economics and mathematical psychology, see the survey by Fishburn (1999); see also Davidson and Marschak (1959), Block and Marschak (1960) and Luce and Suppes (1965).

My theory of random consumer demand is a random choice theory, so the primitive concept is that of a random choice model, which gives a probability distribution on each possible choice set.

I now digress to discuss the literature on the relationship between random choice models and random preference models—not because it is relevant to this research, but rather to persuade readers that it is not. The literature (see Falmagne 1978; Cohen 1980; Barberá and Pattanaik 1986 and McFadden and Richter 1990) concerns conditions on random choice models to be rationalizable by (or observationally equivalent to) random preference models. Typically, random preferences are required to be complete and transitive, but are otherwise unconstrained.

As these conditions are not easy to verify, I do not know if there is a random preference rationalization for my random demand. I *do* know that, if random preferences are given by random utilities, with independent utility across objects, as they often are in the discrete choice literature, then there is no such rationalization: I cannot ensure that from every budget non-frontier elements are chosen with probability zero. In any case, I am not in an abstract choice setting, but rather in a consumer demand setting, and one would probably want to impose restrictions such as convexity and monotonicity on random preference rationalizations. This, of course, complicates the problem. The question of whether my random demand can be rationalized by monotone and convex random preferences may be of some interest, but it is beyond the scope of this paper. In particular, I do not seek to justify my approach by appealing to a random preference rationalization. Indeed, I consider the random choice approach to have an important advantage over the random preference approach. While there is plenty of theoretical guidance on what restrictions to impose on random preferences, there is little on what restrictions to impose on their distribution. Without such restrictions, the inferential problem is intractable.

I now define a random choice model and a random consumer demand model. The first definition is similar to that in McFadden and Richter (1990). The second definition specializes the model to a consumer demand environment.

Definition 1. The ordered 4-tuple $(X, \mathscr{B}, \mathscr{C}, p)$ is a *random choice model* if the following hold:

- X is a non-empty set. I will call X the *universe* and its elements *objects*.
- $\mathscr{B}$ is a set of non-empty subsets of X. I will call $(X, \mathscr{B})$ the *budget space* and the elements of $\mathscr{B}$ *budgets*. A budget is interpreted as a set of objects from which a decision maker must choose a single element.
- $\mathscr{C}$ is a function on $\mathscr{B}$ assigning to each budget B an algebra $\mathscr{C}_B$ of subsets of B. For every budget $B \in \mathscr{B}$, a budget subset $C \in \mathscr{C}_B$ is an event of the measurable space $(B, \mathscr{C}_B)$, and this event is interpreted as the choice by the economic agent of some element of C when faced with budget B.
- p is a function on $\mathscr{B}$ assigning to each budget B a finitely additive probability measure p_B on measurable space (B, C_B). I will call p the *random choice function*. In the special case of consumer choice, I will use the term *random demand function*. For every budget $B \in \mathscr{B}$ and every budget subset $C \in \mathscr{C}_B$, $p_B(C)$ is the probability that the decision maker chooses some element of C when faced with budget B.

Definition 2. A *random consumer demand model* is a random choice model $(X, \mathscr{B}, \mathscr{C}, p)$ where

1. for some integer $n \geqslant 2$, $X = \mathbb{R}^n_+$,
2. $\mathscr{B}$ is the set of all non-empty finite subsets of X, and
3. for every budget $B \in \mathscr{B}$, $\mathscr{C}_B$ is the power set of B.

Note that the consumer may face, as a budget, any non-empty finite subset of the consumption set. While widespread in the mathematical psychology literature, the finiteness of budgets goes against the grain of consumer theory, where the consumer faces budgets of the form $\{x \in X: w \cdot x \leqslant m\}$, where w is a vector of positive prices of the n goods and m is the consumer's non-negative income. These *classical* budgets are not finite, and therefore are not in our budget space.

However, we may consider finite lattices of points in classical budgets of whatever density we like, so this is not a serious restriction. Real consumers and econometricians have only a finite set of numbers available to express the quantities of goods they demand or observe, respectively. Furthermore, the currency used in transactions is not infinitely divisible.

Random choice models feature a probability distribution for every budget. For all but the simplest budget spaces, this is a large amount of information, offering an excessive number of degrees of freedom. In practice, we must add discipline to these models by introducing assumptions jointly constraining these distributions. This is done in the next section.

II. Assumptions on p

We move on to assumptions about the random demand function p. Throughout this section, we shall suppose that $(X, \mathscr{B}, \mathscr{C}, p)$ is a random consumer demand model.

The assumptions are mostly analogous to assumptions in classical consumer demand theory. Although they apply to different primitives than the classical axioms do, they are, in a sense made precise in Theorems C1 and C2, weaker than the classical axioms. These results, together with the general acceptance of the classical axioms, are justification for the assumptions that follow.

Assumptions on p are classified as either assumptions on binary choice or assumptions relating binary and multiple choice.

Binary choice probabilities

The assumptions on binary choice probabilities are analogous to assumptions about binary preferences in deterministic theories. The following notation is a handy and commonly used shorthand for binary choice probabilities.

Definition 3. For every x, $y \in X$ such that $x \neq y$, define $p(x, y) \equiv p_{\{x,\ y\}}(\{x\})$.

In deterministic theories of choice, binary preferences are usually complete. A probability distribution on a singleton set must assign probability 1 to that set, and for all distinct x, $y \in X$, $p(x, y) + p(y, x) = 1$. The probabilistic framework thus builds in conditions analogous to completeness.

In a survey of stochastic utility, Fishburn (1999) describes nine stochastic analogues of the transitivity assumption, including the following.

Assumption 1. (Moderate Stochastic Transitivity). For every x, y, $z \in X$ such that $x \neq y$, $y \neq z$, and $x \neq z$, $\min(p(x,y), p(y,z)) \geqslant \frac{1}{2} \Rightarrow p(x,z) \geqslant \min(p(x,y), p(y,z))$.

We now turn to analogues of assumptions on preferences in consumer demand theory in particular. The next assumption is analogous to monotonicity, expressing the idea that more is better than less. The stochastic nature of human choice is often attributed to conflict in choice. When distinct objects x and y are such that $x_i \geqslant y_i$ for $i = 1, \ldots, n$, there is no conflict in choice: one object is unambiguously better in at least one dimension, and no worse in every dimension. The assumption states that x will invariably be chosen from the budget $\{x, y\}$.

Assumption 2. (Monotonicity). For every x, $y \in X$ such that $x \neq y$, $x \geqslant y \Rightarrow p(x, y) = 1$.

The next assumption is analogous to the classical convexity assumption, which expresses the idea that, to obtain more and more of one good, the consumer is less and less willing to forgo other goods. The following assumption expresses the similar idea that, as one moves through the consumption set in any direction, the strength of the propensity to choose a farther element over a nearer one does not increase.

Assumption 3. (Convexity). For every x, $y \in X$, $p(\frac{1}{2}x + \frac{1}{2}y, x) \geqslant p(y, \frac{1}{2}x + \frac{1}{2}y)$.

The final assumption on binary preferences states that, whenever there is conflict between two choices, that is whenever one choice dominates the other in one dimension and is dominated in another, choice probabilities are non-degenerate. The assumption serves to connect tripleton budgets in such a way that there is a single L-utility scale on the consumption set rather than a collection of local scales.

Assumption 4. (Non-Degenerate Choice in Cases of Conflict). For every x, $y \in X$ such that there are i and j with $x_i > y_i$ and $x_j < y_j$, $p(x, y) \in (0, 1)$.

Luce (1959), in a more general context, uses Strong Stochastic Transitivity (SST: replace 'min' with 'max' in Assumption 1, Moderate Stochastic Transitivity) to deliver a single L-utility scale. SST has undesirable implications in our consumer demand context. To see this, suppose that (1, 2) and (2, 1) are chosen with equal probability from {(1, 2), (2, 1)}. Let $\varepsilon > 0$. Assumption 2 (Monotonicity) requires that (2, 1) be chosen with probability 1 from $\{(2, 1), (2 - \varepsilon, 1)\}$, and SST then implies that (1, 2) is chosen with probability 1 from $\{(1, 2), (2 - \varepsilon, 1)\}$. Assumption 4 allows us to obtain a global scale without assuming SST. Note that Assumption 1, Moderate Stochastic Transitivity, only constrains the probability of choosing (1, 2) from $\{(1, 2), (2 - \varepsilon)\}$ to be greater than 1/2, which is reasonable.

We will see that Assumption 4 is indispensable, in the sense that any random demand behaviour represented by a regular L-utility function satisfies this assumption, an immediate consequence of Theorem 2.

Multiple choice probabilities

The assumption relating binary choice probabilities and multiple choice probabilities is analogous to the usually implicit assumption, known as menu independence, that preferences do not depend on budgets. The assumption, like menu independence, is important for obtaining a representation (utility or L-utility, as the case may be) that is a single function on the consumption set.

The assumption will lead to independence of irrelevant alternatives on budget frontiers, a property that is considered by many, with justification, to be too strong for discrete choice models in which the universe of objects is abstract and unstructured. So it is important to make two observations. First, menu independence has been generally accepted in the context of classical consumer demand for decades. Second, the theory of random demand developed here is a near generalization of classical demand, in the sense of Theorems C1 and C2.

The following assumption, due to Luce (1959), constrains choice distributions across budgets. In particular, it relates binary choice probabilities to multiple choice probabilities. It is the key assumption allowing the representation of random consumer demand by a single function on the consumption set.

The first part of the axiom concerns budget sets for which all choices on binary subsets have non-zero probability. For such budgets and their subsets, the axiom states that relative choice probabilities are independent of the presence of other alternatives in the budget: for every budget B and non-empty $C \subseteq B$, the distribution $p_C(\cdot)$ coincides with the conditional distribution $p_B(\cdot|C)$ on C. Luce calls this part of the axiom 'a probabilistic version of ... [Arrow's] independence-from-irrelevant-alternatives idea'.

The second part of the axiom concerns budget sets for which some choices on binary subsets have zero probability. It says that in a budget B with elements x and y satisfying $p(x, y) = 0$, x may be ignored: the probability of choosing x from B is zero, and the probability of choosing another element from B is the same as the probability of choosing it from $B \backslash \{x\}$.

Assumption 5. (Luce's Choice Axiom). For every $B \in \mathscr{B}$, and every $S \subseteq B$,

1. If $p(x, y) \in (0, 1)$ for every $x, y \in B$ such that $x \neq y$, then for every $R \subseteq S$,

$$p_B(R) = p_S(R) \cdot p_B(S).$$

2. If $p(x, y) = 0$ for some $x, y \in B$ such that $x \neq y$, then

$$p_B(S) = p_{B\backslash\{x\}}(S\backslash\{x\}).$$

We will see in Appendix A that Luce's Choice Axiom introduces further restrictions on binary choice. It implies a stronger condition on $p(x, y)$, $p(y, z)$ and $p(x, z)$ than Moderate Stochastic Transitivity whenever these probabilities are all in (0, 1).

I now explain why I rule out uncountable budget sets such as classical budgets. Allowing both uncountable budget sets and the finite budget sets that appear in many of our assumptions is problematic. Suppose choice probabilities on uncountable sets are absolutely continuous with respect to Lebesgue measure, a reasonable case to consider in the context of consumer demand, if not the only one. Then part one of Luce's choice axiom fails to constrain when R is finite and B is uncountable, whatever the cardinality of S.

III. Representation Theorems

The two theorems of this section concern the representation of random demand functions by regular functions on the consumption set. The following definition of regularity is specific to this paper.

Definition 4. A function $u : \mathbb{R}\backslash\{0\} \rightarrow \mathbb{R}$, where $n \geqslant 2$, is regular if

1. u is non-decreasing, and
2. for every $w \in \mathbb{R}^n_{++}$ and every $m \in \mathbb{R}_{++}$, u is log-concave on the classical budget frontier $\{x \in X : w \cdot x = m\}$.

Together, the theorems identify theoretically consistent random demand with a regular L-utility function, which is important for applied consumer demand analysis: the econometrician can work with L-utility functions, rather than random demand functions themselves. In a companion paper (McCausland 2004), I have developed a practical method for Bayesian semi-nonparametric inference for L-utility functions using consumer demand data.

The first theorem establishes the existence and uniqueness of the representation up to a multiplicative positive constant. The significance of existence is that the econometrician can work with L-utility functions and not miss any theoretically consistent random demand. The significance of uniqueness (the representation can be made unique by fixing the value of the function at any one point) is that the L-utility function is identified by the data.

We define, for each budget B, the frontier $\hat{B}$ as the subset of objects not vector-dominated by other elements of B. We will see that $\hat{B}$ is the set of objects chosen with non-zero probability from B.

Definition 5. For any budget $B \in \mathscr{B}$, define $\hat{B}$, the frontier of B, by

$$\hat{B} = \{x \in B : \text{ there is no } y \in B\backslash\{x\} \text{ such that } y \geqslant x\}.$$

We are now ready to state the first theorem.

Theorem 1. (Existence and Uniqueness of Representation). If $(X, \mathscr{B}, \mathscr{C}, p)$ is a random consumer demand model satisfying Assumptions 1, 2, 3, 4 and 5, then there exists a regular function $u : X\backslash\{0\} \rightarrow \mathbb{R}_{++}$, unique up to the multiplication of a positive constant, such that, for every budget $B \in \mathscr{B}$ and every event $C \in \mathscr{C}_B$, $p_B(C)$ is given by

$$(1) \qquad p_B(C) = \sum_{x \in C \cap \hat{B}} u(x) \Big/ \sum_{y \in \hat{B}} u(y).$$

As in deterministic theories of consumer demand, we have a representation for theoretically consistent demand as a single function on the classical consumption set. Note that demand is concentrated on the budget frontier, and that on this frontier bundles with higher L-utility are chosen with greater probability than bundles with lower L-utility. The log-concavity properties of L-utility are analogous to the quasi-concavity of utility.

The next definition establishes the relational symbols $\not\asymp$ and $\asymp$ as shorthand notation denoting whether or not a pair of objects in X features one object vector-dominating the other. The importance of this notation lies in the consequence of Assumptions 2 (Monotonicity) and 4 (Non-Degenerate Choice in Cases of Conflict) that, for all $x, y \in X$, the choice from $\{x, y\}$ is non-degenerate (i.e. $p(x, y) \in (0, 1)$) if and only if $x \asymp y$.

Definition 6. Define the binary relation $\asymp$ on X by $x \asymp y \Leftrightarrow x \not\geqq y$ and $y \not\geqq x$, and let the binary relation $\not\asymp$ on X denote its complement.

The second theorem establishes the completeness of the representation. The significance of completeness is that an L-utility function estimated by an econometrician is guaranteed to be the representation of *some* theoretically consistent random demand behaviour.

Theorem 2. (Completeness of Representation). Suppose that $n \geqslant 2$ and that $u : \mathbb{R}^n_+\backslash\{0\} \rightarrow \mathbb{R}_{++}$ is regular. Then there exists a unique random consumer demand model $(\mathbb{R}^n_+, \mathscr{B}, \mathscr{C}, p)$ such that

1. $(\mathbb{R}^n_+, \mathscr{B}, \mathscr{C}, p)$ satisfies Assumptions 1, 2, 3, 4 and 5; and
2. for all budgets $B \in \mathscr{B}$, and all events $C \in \mathscr{C}_B$, $p_B(C)$ is given by (1).

IV. Conclusions

I have drawn from stochastic theories of choice in mathematical psychology and deterministic theories of choice and consumer demand in economics to develop a new theory of random consumer demand. The theory has several desirable properties that motivate its application to empirical consumer demand problems.

The representation theorems establish an identification of any theoretically consistent random demand function with a regular L-utility function and vice versa. An econometrician can therefore work with regular L-utility functions rather than with random demand functions directly. This is more convenient.

The theory is intrinsically stochastic, which allows econometricians to apply the theory without adding extrinsic randomness in the form of residuals.

The theory does not stand or fall on a sharp testable implication such as the Strong Axiom of Revealed Preference. There are degrees of fit, and the theory can be evaluated on this basis. The theory measures the ‘fit’ of an observed choice to a regular-utility function by the relative L-utilities of the choice and its feasible alternatives. This is an intrinsic measure of fit, and stands in contrast to extrinsic measures of fit, such as the Euclidean distance of a choice to the object that maximizes the utility function.

While random demand functions have the 'independence of irrelevant alternatives' property on budget frontiers and their subsets—a property many consider too restrictive in the context of the discrete choice literature, where choice sets are abstract and unstructured—the theory is a near generalization, in a sense made precise in Theorems C1 and C2, of classical demand theory, which is generally accepted.

In a related paper I have described a practical method for Bayesian inference using the theory (McCausland 2004). I use polynomials on a transformed consumption set to approximate log L-utility functions, giving flexibility and regularity on a large subset of the consumption set. I apply the theory and inferential methods to analyse data from a consumer experiment.

APPENDIX A: PROOF OF EXISTENCE AND UNIQUENESS OF REPRESENTATION

Let $(X, \mathscr{B}, \mathscr{C}, p)$ be a random consumer demand model satisfying Assumptions 1, 2, 3, 4 and 5.

The proof proceeds as follows. First I prove a useful triplet result. Then I construct a function u on $X \backslash \{0\}$. I next show that, for all $x, y \in X$ such that $x \asymp y$ (i.e. neither $x \geqslant y$ nor $y \geqslant x$), we can use u to reconstruct the binary choice probability $p(x, y)$. I then use this result to show that for all budgets B we can use u to reconstruct the choice distribution p_B. Next, I show that u is regular. Finally, I show that u is unique up to the multiplication of a positive constant.

A triplet result

The following triplet result is a useful intermediate result.

Claim A1. For all objects $x, y, z \in X$ satisfying $x \asymp z$, $y \asymp z$ and $z \asymp x$,

$$p(x, y)p(y, z)p(z, x) = p(y, x)p(z, y)p(x, z). \tag{2}$$

Proof. Let $x, y, z \in X$ satisfy $x \asymp z$, $y \asymp z$and $z \asymp x$. Let budget $B \equiv \{x, y, z\}$. Assumptions 4 (Non-Degenerate Choice in Cases of Conflict) and 5 (Luce's Choice Axiom) (part 1) give us the following six equations:

$$p_B(\{x, z\}) \cdot p(x, z) = p_B(\{x\}) = p_B(\{x, y\}) \cdot p(x, y)$$

$$p_B(\{x, y\}) \cdot p(y, x) = p_B(\{y\}) = p_B(\{y, z\}) \cdot p(y, z)$$

$$p_B(\{y, z\}) \cdot p(z, y) = p_B(\{z\}) = p_B(\{x, z\}) \cdot p(z, x)$$

and guarantee that all binary probabilities in the six equations are non-zero. Since $\{x, y\} \cup \{x, z\} = B$, $p_B(\{x, y\}) + p_B(\{x, z\}) \geqslant 1$, $p_B(\{x, y\})$ and $p_B(\{x, z\})$ cannot both be zero, and therefore $p_B(\{x\}) > 0$. Therefore $p_B(\{x, y\}) > 0$ and $p_B(\{x, z\}) > 0$. Similarly, all the probabilities in the second and third lines must also be non-zero.

We thus obtain

$$1 = \frac{p_B(\{x\})}{p_B(\{y\})} \cdot \frac{p_B(\{y\})}{p_B(\{z\})} \cdot \frac{p_B(\{z\})}{p_B(\{x\})} = \frac{p(x, y)}{p(y, x)} \cdot \frac{p(y, z)}{p(z, y)} \cdot \frac{p(z, x)}{p(x, z)}$$

and equation (2) immediately follows. $\square$

Construction of u

I now construct my representation. This resembles the representation of Theorem 4 in Luce (1959). Furthermore, the approach I use to construct the representation is similar to that of Luce.

However, the two additional assumptions that Luce makes to guarantee the uniformity of his representation across budgets are different from the key assumption that I use to guarantee this uniformity, i.e. Assumption 4 (Non-Degenerate Choice on Budget Frontiers). This assumption neither implies nor is implied by Luce's two additional assumptions.

Choose an object $a \in X$ such that $a > 0$, and a real constant $k > 0$. Define $u(a) \equiv k$. Now consider any object $b \in X \backslash \{0, a\}$. If $b \asymp a$, then $p(a, b) > 0$ and $p(b, a) > 0$ by Assumption 4. I define

$$u(b) \equiv k \cdot \frac{p(b,a)}{p(a,b)},$$

and note that it must be positive.

If $b \not\asymp a$, then let $X_{ab} \equiv \{x \in X : x \asymp a \text{ and } x \asymp b\}$, and define $u_{ab} : X_{ab} \to \mathbb{R}$ by

$$u_{ab}(x) = k \cdot \frac{p(b,x)}{p(x,b)} \cdot \frac{p(x,a)}{p(a,x)} \quad \forall x \in X_{ab}.$$

I will show that $X_{ab} \neq \varnothing$ and that u_{ab} is well defined, positive and constant. I will then define $u(b)$ to be this constant value.

We can express X_{ab} as the following union of rectangles:

$$X_{ab} = \bigcup_{\substack{i,j \in \{1,\ldots,n\} \\ i \neq j}} X_{ab}^{\{i,j\}} \equiv \bigcup_{\substack{i,j \in \{1,\ldots,n\} \\ i \neq j}} \{x \in X : x_i < \underline{x}_i \text{ and } x_j > \overline{x}_j\},$$

where

$$\underline{x} \equiv \begin{cases} a & a \leqslant b \\ b & a \geqslant b \end{cases} \quad \text{and} \quad \bar{x} \equiv \begin{cases} b & a \leqslant b \\ a & a \geqslant b. \end{cases}$$

(Since $b \not\asymp a$, it must be the case that either $a \geqslant b$ or $b \geqslant a$.)

Since $a > 0$ and $b \neq 0$, at least one of the rectangles $X_{ab}^{\{i,j\}}$ is non-empty, and so $X_{ab} \neq \varnothing$. Assumption 4 guarantees that, for all $x \in X_{ab}$, $p(x, a)$, $p(a, x)$, $p(x, b)$, and $p(b, x)$ are all positive. Therefore $u_{ab}(x)$ is well defined and positive for all $x \in X_{ab}$.

I now show that u_{ab} is constant on X_{ab}. I take arbitrary elements x, $y \in X_{ab}$, and show that $u_{ab}(x) = u_{ab}(y)$. I consider the cases $x \asymp y$ and $x \not\asymp y$ separately. First, suppose x, $y \in X_{ab}$ and $x \asymp y$. We can apply the triplet result twice to obtain

$$\frac{p(x,a)}{p(a,x)} \cdot \frac{p(a,y)}{p(y,a)} = \frac{p(x,y)}{p(y,x)} = \frac{p(x,b)}{p(b,x)} \cdot \frac{p(b,y)}{p(y,b)}$$

$$k \cdot \frac{p(b,x)}{p(x,b)} \cdot \frac{p(x,a)}{p(a,x)} = k \cdot \frac{p(b,y)}{p(y,b)} \cdot \frac{p(y,a)}{p(a,y)}$$

$$u_{ab}(x) = u_{ab}(y)$$

Now suppose x, $y \in X_{ab}$ and $x \not\asymp y$. Objects x and y must be in the same rectangle $X_{ab}^{\{i,j\}}$, since otherwise $x \asymp y$. I now construct a $z \in X_{ab}$ such that $x \asymp z$ and $y \asymp z$, so that $u_{ab}(z)$ is equal to both $u_{ab}(x)$ and $u_{ab}(y)$. Define $z \equiv (z_1, \ldots, z_n) \in X_{ab}$ as follows:

$$z_k = \begin{cases} \frac{1}{2}\underline{x}_i + \frac{1}{2}\max(x_i, y_i) & k = i \\ \frac{1}{2}\bar{x}_j + \frac{1}{2}\min(x_j, y_j) & k = j \\ 0 & k \in \{1, \ldots, n\} \backslash \{i, j\}. \end{cases}$$

Since $x \asymp z$ and $y \asymp z$, $u_{ab}(x) = u_{ab}(z) = u_{ab}(y)$.

The cases $x \asymp y$ and $x \not\asymp y$ are exhaustive, so I have shown that u_{ab} is constant and positive on the non-empty set X_{ab}. I now define $u(b)$ to be this constant value.

Since b was an arbitrary element of $X \backslash \{a, 0\}$, I have constructed a function u on the entire set $X \backslash \{0\}$.

Non-degenerate choices on binary budgets

I now show that, for all x, $y \in X$ such that $x \asymp y$ (i.e. neither $x \geqslant y$ nor $y \geqslant x$), we can use u to reconstruct the binary choice probability $p(x, y)$.

Claim A2. For every x, $y \in X$ such that $x \asymp y$,

$$\frac{p(x,y)}{p(y,x)} = \frac{u(x)}{u(y)}. \tag{3}$$

Proof. Let $a \in X$ be the object used in Appendix A to construct u. Let $x, y \in X \backslash \{0\}$ be such that $x \asymp y$. Choose i, $j \in \{1, \ldots, n\}$ such that $y_i > 0$ and $j \neq i$.

First I show that if $x_i > 0$ then (3) holds. Then I use this result to show that (3) holds even if $x_i = 0$.

Suppose $x_i > 0$, and define $z \equiv (z_1, \ldots, z_n)$ by

$$z_k = \begin{cases} \max(x_j, y_j, a_j) + 1 & k = j \\ 0 & k \in \{1, \ldots, n\} \backslash \{j\}. \end{cases}$$

Then $z \asymp x$, $z \asymp y$, and $z \asymp a$. Using the triplet result (Claim A1) and the definition of u, we obtain

$$\frac{p(x,y)}{p(y,x)} = \frac{p(x,z)}{p(z,x)} \cdot \frac{p(z,y)}{p(y,z)} = u(x) \cdot \frac{p(a,z)}{p(z,a)} \cdot \left[u(y) \cdot \frac{p(a,z)}{p(z,a)}\right]^{-1} = \frac{u(x)}{u(y)}.$$

Now suppose $x_i = 0$. Let $w = \frac{1}{2}x + \frac{1}{2}y$. Then $w_i > 0$, $w \asymp x$, and $w \asymp y$, and so, by the result just proved,

$$\frac{p(x,w)}{p(w,x)} = \frac{u(x)}{u(w)} \quad \text{and} \quad \frac{p(y,w)}{p(w,y)} = \frac{u(y)}{u(w)}.$$

By the triplet result (Claim A1),

$$\frac{p(x,y)}{p(y,x)} = \frac{p(x,w)}{p(w,x)} \cdot \frac{p(w,y)}{p(y,w)} = \frac{u(x)/u(w)}{u(y)/u(w)} = \frac{u(x)}{u(y)}. \quad \square$$

Choices on finite budgets

I now use the previous result to show that for every budget B we can use u to reconstruct the choice distribution p_B.

Claim A3. For every budget $B \in \mathscr{B}$, and every event $C \in \mathscr{C}_B$,

$$p_B(C) = \sum_{x \in C \cup \hat{B}} u(x) \Big/ \sum_{y \in \hat{B}} u(y),$$

where $\hat{B}$ is the budget frontier (Definition 5) of B.

Proof. Let $B \in \mathscr{B}$. Repeated application of Assumption 5 (Luce's Choice Axiom, part 2), gives $p_B(x) = p_{\hat{B}}(x)$ for all $x \in \hat{B}$, and, since p_B is a probability measure, $p_B(x) = 0$ for all $x \in B \backslash \hat{B}$.

Using the result on binary budgets (Claim A2) and part 1 of Assumption 5 (Luce's Choice Axiom), we have, for every $x \in \hat{B}$,

$$\frac{1}{p_{\hat{B}}(x)} = \frac{\sum_{y\in\hat{B}} p_{\hat{B}}(y)}{p_{\hat{B}}(x)} = \sum_{y\in\hat{B}} \frac{p_{\hat{B}}(y)}{p_{\hat{B}}(x)} = 1 + \sum_{y\in\hat{B}\setminus\{x\}} \frac{p_{\hat{B}}(\{x,y\})p(y,x)}{p_{\hat{B}}(\{x,y\})p(x,y)}$$
$$= 1 + \sum_{y\in\hat{B}\setminus\{x\}} \frac{u(y)}{u(x)} = \frac{\sum_{y\in\hat{B}} u(y)}{u(x)}$$

Therefore for every $x \in B$,

$$p_B(x) = \begin{cases} u(x)/\sum_{y\in\hat{B}} u(y) & x\in\hat{B} \\ 0 & x\in B\setminus\hat{B}. \end{cases}$$

Since p_B is a probability measure, the claim follows. $\square$

Regularity of u

The following results establish the regularity of u.

Claim A4. The function u is non-decreasing.

Proof. Let $x, y \in X\setminus\{0\}$satisfy $x \geqslant y$. Note that $p(x, y) = 1$ by Assumption 2 (Monotonicity). Let $z \in X\setminus\{0\}$ satisfy $z \asymp x$ and $z \asymp y$.

I first show that $p(z, y) \geqslant p(z, x)$.

- Case $p(z,x) \geqslant \frac{1}{2}$: Apply Assumption 1 (Moderate Stochastic Transitivity) to obtain $p(z, y) \geqslant \min(p(z, x), p(x, y)) \geqslant p(z, x)$.
- Case $p(z,x) \geqslant \frac{1}{2}$: Suppose to the contrary that $p(z, y) < p(z, x)$. Then $p(y,z) > p(x,z) > \frac{1}{2}$. By Assumption 1 (Moderate Stochastic Transitivity), we obtain $p(x, z) \geqslant \min(p(x, y), p(y, z)) > p(y, z)$, and therefore $p(z, x) < p(z, y)$, which contradicts $p(z, y) < p(z, x)$.

Since $p(z, y) \geqslant p(z, x)$, $u(x) \geqslant u(y)$. So we have $x \geqslant y \Rightarrow u(x) \geqslant u(y)$ for all $x, y \in X\setminus\{0\}$. That is, u is non-decreasing. $\square$

Claim A5. For every $w \in \mathbb{R}^n_{++}$ and every $m \in \mathbb{R}_{++}$, u is concave on the classical budget frontier $\{z \in X{:}w\cdot z = m\}$.

Proof. Let $w \in \mathbb{R}^n_{++}$ and $m \in \mathbb{R}_{++}$. Let x, $y \in \{z \in X{:}w\cdot z = m\}$. By Assumption 3 (Convexity),

$$p\left(\frac{1}{2}x + \frac{1}{2}y, x\right) \geqslant p\left(y, \frac{1}{2}x + \frac{1}{2}y\right) \tag{4}$$

and

$$p\left(\frac{1}{2}x + \frac{1}{2}y, y\right) \geqslant p\left(x, \frac{1}{2}x + \frac{1}{2}y\right). \tag{5}$$

It must be the case that $x \asymp \frac{1}{2}x + \frac{1}{2}y \asymp y$, since otherwise $w'x \neq w'y$. Therefore the probabilities in equations (4) and (5) are all non-zero, and so

$$\frac{p(\frac{1}{2}x+\frac{1}{2}y, x)}{p(x, \frac{1}{2}x+\frac{1}{2}y)} \geqslant \frac{p(y, \frac{1}{2}x+\frac{1}{2}y)}{p(\frac{1}{2}x+\frac{1}{2}y, y)}$$

$$\frac{u(\frac{1}{2}x+\frac{1}{2}y)}{u(x)} \geqslant \frac{u(y)}{u(\frac{1}{2}x+\frac{1}{2}y)}$$

$$\left[u\left(\frac{1}{2}x+\frac{1}{2}y\right)\right]^2 \geqslant u(x)u(y)$$

$$\log u\left(\frac{1}{2}x+\frac{1}{2}y\right) \geqslant \frac{1}{2}\log u(x) + \frac{1}{2}\log u(y).$$

Since this is true for all distinct x, y on the classical budget frontier, u must be log-concave there. □

Uniqueness of u

Claim A6. The representation u is unique up to the multiplication of a positive constant.

Proof. We want to show that if u and u' are both regular, and both represent $(X, \mathscr{B}, \mathscr{C}, p)$, then there exists a constant $c>0$ such that $u' = cu$. Suppose u and u' are both regular, and both represent $(X, \mathscr{B}, \mathscr{C}, p)$. Let $z \in X\backslash\{0\}$ satisfy $z>0$. Let $c \equiv u'(z)/u(z)$.

Now choose any $x \in X\backslash\{0\}$. If $x = z$, then $u'(x) = cu(x)$ immediately. If $x \neq z$, then choose a $y \in X_{zx}$. Then

$$\frac{u(x)}{u(y)} = \frac{p(x,y)}{p(y,x)} = \frac{u'(x)}{u'(y)} \quad \text{and} \quad \frac{u(y)}{u(z)} = \frac{p(y,z)}{p(z,y)} = \frac{u'(y)}{u'(z)},$$

and therefore

$$u'(x) = u(x)\cdot\frac{u'(y)}{u(y)} = u(x)\cdot\frac{u'(z)}{u(z)} = cu(x). \quad \square$$

APPENDIX B: PROOF OF COMPLETENESS OF REPRESENTATION

Let $n \geqslant 2$, and let $u : \mathbb{R}^n_+\backslash\{0\} \to \mathbb{R}_{++}$ be any regular function. Let $X = \mathbb{R}^n_+$ and $\mathscr{B}$ be the set of finite subsets of X. For each $B \in \mathscr{B}$, let $\mathscr{C}$ be the power set of B. I first use u to construct a p, and note that u represents p. I then show that $(X, \mathscr{B}, \mathscr{C}, p)$ satisfies Assumptions 1, 2, 3, 4 and 5.

Construct p so that, for every $B \in \mathscr{B}$ and every $C \in \mathscr{C}_B$,

$$p_B(C) \equiv \sum_{x \in C \cap \hat{B}} u(x) \Big/ \sum_{y \in \hat{B}} u(y).$$

Clearly, p is uniquely specified, and u represents it.

Claim B1. $(X, \mathscr{B}, \mathscr{C}, p)$ satisfies Assumption 1 (Moderate stochastic transitivity).

Proof. Let $x, y, z \in X$ satisfy $x \neq y$, $y \neq z$ and $x \neq z$, and suppose $p(x,y) \geqslant \frac{1}{2}$ and $p(y,z) \geqslant \frac{1}{2}$. If $x \geqslant y$, then $u(x) \geqslant u(y)$ by monotonicity of u. If $x \asymp y$, then $p(x, y) = u(x)/[u(x) + u(y)]$ and so $u(x) \geqslant u(y)$. We can rule out $y \geqslant x$ since it contradicts $p(x,y) \geqslant \frac{1}{2}$. Therefore $u(x) \geqslant u(y)$. Similarly, $u(y) \geqslant u(z)$.

We show the result that $p(x, z) \geqslant \min(p(x, y), p(y, z))$ for three cases.

- Case $x \geqslant z$: Then $p(x, z) = 1 \geqslant \min(p(x, y), p(y, z))$ and we are done.
- Case $z \geqslant x$: Since $u(x) \geqslant u(y) \geqslant u(z)$, monotonicity of u rules out this case.
- Case $x \asymp z$: Then $p(x, z)/p(z, x) = u(x)/u(z)$. Since $x \geqslant y$ and $y \geqslant z$ imply $x \geqslant z$, at least one of $x \asymp y$ and $y \asymp z$ must hold. If $x \asymp y$, then $p(x, y)/p(y, x) = u(x)/u(y)$, and therefore

$$\frac{p(x,z)}{p(z,x)} = \frac{u(x)}{u(z)} \geqslant \frac{u(x)}{u(y)} = \frac{p(x,y)}{p(y,x)} \geqslant \min\left(\frac{p(x,y)}{p(y,x)}, \frac{p(y,z)}{p(z,y)}\right).$$

If $y \asymp z$, then $p(y, z)/p(z, y) = u(y)/u(z)$, and therefore

$$\frac{p(x,z)}{p(z,x)} = \frac{u(x)}{u(z)} \geqslant \frac{u(y)}{u(z)} = \frac{p(y,z)}{p(z,y)} \geqslant \min\left(\frac{p(x,y)}{p(y,x)}, \frac{p(y,z)}{p(z,y)}\right).$$

Either way, since the transformation $f(p) = p/(1-p)$ is monotonically increasing, $p(x, z) \geqslant \min(p(x, y), p(y, z))$. □

Claim B2. $(X, \mathscr{B}, \mathscr{C}, p)$ satisfies Assumptions 2 (Monotonicity) and 4 (Non-Degenerate Choice in Cases of Conflict).

Proof. Let $x, y \in X$ satisfy $x \neq y$. If $x \geqslant y$, then $p(x, y) = p_{\{x\}}(\{x\}) = 1$. If $y \geqslant x$, then $p(x, y) = 0$. If $x \asymp y$, then $p(x, y) = u(x)/[u(x) + u(y)] \in (0, 1)$. □

Claim B3. $(X, \mathscr{B}, \mathscr{C}, p)$ satisfies Assumption 3 (Convexity).

Proof. Let $x, y \in X$ satisfy $x \asymp y$. Let $z \equiv \frac{1}{2}x + \frac{1}{2}y$. Since x, y, and z lie on a budget frontier, condition 2 of Theorem 1 gives

$$\log u(z) \geqslant \frac{1}{2}\log u(x) + \frac{1}{2}\log u(y)$$

$$\frac{1}{2}\log u(z) - \frac{1}{2}\log u(y) \geqslant \frac{1}{2}\log u(x) - \frac{1}{2}\log u(z)$$

$$u(z)/u(y) \geqslant u(x)/u(z).$$

Now since $x \asymp z$ and $y \asymp z$,

$$p(z,x) = u(z)/[u(x) + u(z)] = 1/[1 + u(x)/u(z)]$$

$$p(y,z) = u(y)/[u(y) + u(z)] = 1/[1 + u(z)/u(y)],$$

and therefore $p(z, x) \geqslant p(y, z)$. □

Claim B4. $(X, \mathscr{B}, \mathscr{C}, p)$ satisfies Assumption 5 (Luce's Choice Axiom).

Proof. Let $B \in \mathscr{B}$, and let $S \subseteq B$. First take the case that $p(x, y) \in (0, 1)$ for every $x, y \in B$ such that $x \neq y$. Then $\hat{B} = B$ and $\hat{S} = S$. Let $R \subseteq S$. Then

$$p_B(R) = \sum_{x \in R} u(x) \Big/ \sum_{y \in B} u(y),$$

$$p_S(R) = \sum_{x \in R} u(x) \Big/ \sum_{y \in S} u(y),$$

$$p_B(S) = \sum_{x \in S} u(x) \Big/ \sum_{y \in B} u(y),$$

and clearly, $p_B(R) = p_S(R) \cdot p_B(S)$.

Now take the case where there exists $x, y \in B$ such that $x \neq y$ and $p(x, y) = 0$. Then $x \notin \hat{B}$ and $\hat{B} = \widehat{B \setminus \{x\}}$ and

$$p_B(S) = \sum_{x \in S \cap \hat{B}} u(x) \Big/ \sum_{y \in \hat{B}} u(y) = \sum_{x \in S/\{x\} \cap \widehat{B \setminus \{x\}}} u(x) \Big/ \sum_{y \in \widehat{B \setminus \{x\}}} u(y)$$

$$= p_{B \setminus \{x\}}(S \setminus \{x\}).$$

APPENDIX C: TWO THEOREMS ON RANDOM DEMAND AS A NEAR GENERALIZATION OF CLASSICAL DEMAND

The following two theorems elucidate the relationship between the random demand introduced in this paper and classical demand. A major purpose of these theorems is to justify our assumptions, particularly Assumption 5, Luce's choice axiom. Theoretically consistent random demand functions have the independence of irrelevant alternatives (IIA) property on budget frontiers and their subsets, a property that many, starting with Debreu (1960), have criticized for its implausible implications in certain choice contexts. In the discrete choice literature, where choice sets are usually abstract and unstructured, IIA is usually considered, with justification, as being too restrictive.

However, menu independence, to which Luce's Choice Axiom is analogous, is generally accepted in classical demand theory. In the sense made precise by the two following theorems, we are generalizing, not restricting, this theory.

The first theorem shows us that, for any classical demand behaviour, there exists random demand behaviour satisfying all but one of the axioms. The missing axiom is dealt with in the second theorem.

We first need some definitions. Let $X = \mathbb{R}^n_+$, the classical consumption set and let $(X, \mathscr{B})$ be the space of all finite budgets on X.

Define, for any preference relation $\succeq$ on X, the *demand correspondence* $h_{\succeq} : \mathscr{B} \to X$ as follows. For each budget $B \in \mathscr{B}$,

$$h(B) = \{x \in B : \text{there is no } y \in B \text{ such that } y \succeq x \text{ and not } x \succeq y\}.$$

Define, for any random demand function p on $(X, \mathscr{B})$, the *demand support correspondence* $g{:}\mathscr{B} \to X$ as follows. For each budget $B \in \mathscr{B}$,

$$g_p(B) = \{x \in B : p_B(x) > 0\}.$$

I am now ready to state and prove the first theorem.

Theorem C1. Let $\succeq$ be a complete, transitive, strictly monotone, and convex preference relation on X. Then there exists a random demand function p on $(X, \mathscr{B})$ such that

1. for all $B \in \mathscr{B}$, $g_p(B) = h_{\succeq}(B)$
2. p satisfies Assumptions 1, 2, 3 and 5.

Proof. Let $\succeq$ be a complete, transitive, strictly monotone and convex preference relation on X. Define random demand function p as follows. For all $B \in \mathscr{B}$ and all subsets $S \subseteq B$,

$$P_B(S) = \frac{\#[S \cap h(B)]}{\#[h(B)]}$$

where $\#[A]$ denotes the cardinality of any finite set A. We see that, for all $B \in \mathscr{B}$, $p_B(\cdot)$ is the uniform distribution on $h(B)$. The finiteness of B and the completeness and transitivity of $\succeq$ guarantee that $h(B)$ is not empty.

Note that the definition of p implies that, for all $x, y \in X$, $p(x, y) \in \{0, \frac{1}{2}, 1\}$.

I now show that p satisfies Assumption 1, Moderate Stochastic Transitivity. Let $x, y, z \in X$ and suppose that $p(x,y) \geqslant \frac{1}{2}$ and $p(y,z) \geqslant \frac{1}{2}$. If $p(x,y) = \frac{1}{2}$ and $p(y,z) = \frac{1}{2}$, then $x \succeq y$ and $y \succeq z$ and therefore $x \succeq z$ by transitivity. Therefore $p(x,z) \geqslant \frac{1}{2} = \min[p(x,y), p(y,z)]$. If $p(x,y) = \frac{1}{2}$ and $p(y, z) = 1$, then $x \succeq y$ and $y \succ z$ and therefore $x \succ z$ by transitivity. Therefore $p(x, z) = 1 \geqslant \min[p(x, y), p(y, z)]$. Similarly, if $p(x, y) = 1$ and $p(y,z) = \frac{1}{2}$, or if $p(x, y) = p(y, z) = 1$, then $p(x, z) \geqslant \min[p(x, y), p(y, z)]$.

Assumption 2 (Monotonicity) is straightforward. If $x_i > y_i$ for all $i \in \{1, \ldots, n\}$, then $x \succ y$ and $p(x, y) = 0$.

I finally show that p satisfies Assumption 3 (Convexity). Let $x, y \in X$. If $p(y, \frac{1}{2}x + \frac{1}{2}y) = 0$, then $p(\frac{1}{2}x + \frac{1}{2}y, x) \geqslant p(y, \frac{1}{2}x + \frac{1}{2}y)$ trivially. If $p(y, \frac{1}{2}x + \frac{1}{2}y) = \frac{1}{2}$, then $y \succeq \frac{1}{2}x + \frac{1}{2}y$. Convexity of $\succeq$ rules out $x \succ \frac{1}{2}x + \frac{1}{2}y$, so $p(\frac{1}{2}x + \frac{1}{2}y, x) \geqslant \frac{1}{2} = p(y, \frac{1}{2}x + \frac{1}{2}y)$. If $p(y, \frac{1}{2}x + \frac{1}{2}y) = 1$, then $y \succ \frac{1}{2}x + \frac{1}{2}y$. Convexity of $\succeq$ rules out $x \succeq \frac{1}{2}x + \frac{1}{2}y$, so $p(\frac{1}{2}x + \frac{1}{2}y, x) = 1 = p(y, \frac{1}{2}x + \frac{1}{2}y)$.

I now demonstrate that Part 1 of Assumption 5, Luce's Choice Axiom, holds. Let $B \in \mathscr{B}$ and $S \in \mathscr{B}$ such that $S \subseteq B$, and suppose $p(x, y) \in (0,1)$ for all $x, y \in B$ such that $x \neq y$. Then $x \succeq y$ for all $x, y \in B$. Then for all $R \subseteq B$, $h(R) = R$, and so

$$p_B(R) = \frac{\#[R]}{\#[B]}, \quad p_S(T) = \frac{\#[R]}{\#[S]}, \quad \text{and} \quad p_B(S) = \frac{\#[S]}{\#[B]}.$$

Therefore $p_B(R) = p_S(R) \cdot p_B(S)$.

Finally, I show that Part 2 of Luce's Choice Axiom holds. Let $S, B \in \mathscr{B}$ such that $S \subseteq B$. Suppose there exist $x, y \in B$ such that $x \neq y$ and $p(x, y) = 0$. Then $y \succ x$ and therefore $x \notin h(B)$. By transitivity, $h(B \backslash \{x\}) = h(B)$. Therefore

$$\begin{aligned} p_B(S) &= \frac{\#[S \cap h(B)]}{\#[h(B)]} \\ &= \frac{\#[S \backslash \{x\} \cap h(B \backslash \{x\})]}{\#[h(B \backslash \{x\})]} \\ &= p_{B \backslash \{x\}}(S \backslash \{x\}). \quad \square \end{aligned}$$

The second theorem, which follows, shows us that with the missing axiom imposed we can achieve a sort of limiting observational equivalence. The lack of uniformity over $\mathscr{B}$ (i.e. the dependence of ε below on $\{B_1, \ldots, B_N\}$) is unfortunate. In practice, however, I recommend working with budgets that are the intersection of fine lattices with classical budgets. With, in addition, a large but finite bound on the quantities of each good, we have a finite working budget space.

Theorem C2. Let $\succeq$ be a complete, transitive, continuous strictly monotone, and convex preference relation on X. Let $\{B_1, \ldots, B_N\}$ be a finite subset of the budget space $\mathscr{B}$. Then there exists an $\varepsilon > 0$ and a random demand function p on $(X, \mathscr{B})$ such that

1. For all $B \in \{B_1, \ldots, B_N\}$,

$$p_B(h_{\succeq}(B)) \geqslant 1 - \varepsilon,$$

2. p satisfies Assumptions 1, 2, 3, 4 and 5.

Proof. Lemma 2 of Kannai (1974) states that, for every compact K with non-empty interior, and for every complete continuous, strictly monotone, and convex preference order $\succeq$, there exists a sequence $\{\succeq_m\}$ of complete, continuous monotone and convex preference orders such that

1. for every $m > 0$ there exists a continuous concave utility u_m representing $\succ$ (i.e. $x \succ y$ iff $u_m(x) > u_m(y)$; and
2. for every $x, y \in K$ with $x \succ y$ and every sequence $x_m \to x$, $y_m \to y$ with $x_m, y_m \in K$, there exists an $M > 0$ such that $x_m \succ_m y_m$ for all $m > M$.

Let $A = \cup_{i=1}^{N} B_i$ and $K = \times_{i=1}^{N}[0, \max_{x \in A}(x_i)]$. Let $\{\succeq_m\}$ be a sequence given by Lemma 2 of Kannai (1974).

By the finiteness of N and the budgets $B_1, \ldots, B_N$, there exists an $m > 0$ sufficiently large that for all $x, y \in B$, $x \succ y \Rightarrow x \succ_m y$.

Let

$$\delta \equiv \min\{[u_m(x) - u_m(y)] : B \in \{B_1, \ldots, B_N\}, x \in h_{\succeq}(B), y \in B \backslash h_{\succeq}(B)\},$$

and let $C \equiv \max\{\#[B] : B \in \{B_1, \ldots, B_N\}\}$.

Note that δ must be strictly positive, and let $u = -(1/\delta)\log(\min(\varepsilon, C-1)/C)u_m$. The function u is increasing and concave and therefore regular, so by Theorem 2 there exists a unique random demand function p represented by u that satisfies Assumptions 1, 2, 3, 4 and 5. The multiplicative constant $-(1/\delta)\log(\min(\varepsilon, C-1)/C)$ is finite, positive and sufficiently large that, for all $B \in \{B_1, \ldots, B_N\}$,

$$p_B(h_{\succeq}(B)) = 1 - p_B(B \backslash h_{\succeq}(B)) \geqslant 1 - \varepsilon. \quad \square$$

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