On the Multiplicative Inequality

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Abstract

The multiplicative inequality (MI) introduced by Sattath and Tversky (1976) is a rare example of a simple and intuitively appealing condition relating choice probabilities across choice sets of different sizes. It is also a testable implication of two models of stochastic discrete choice: the Elimination by Aspects model of Tversky (1972b) and the independent random utility model. We prove several results on the multiplicative inequality and its relationship to the regularity condition. One major result illustrates how little the MI constrains binary choice probabilities: it implies that every system of binary choice probabilities on a universe of choice objects can be extended to a complete system of choice probabilities satisfying the MI. In this sense, the MI is complementary to axioms for binary choice probabilities, of which many have been proposed. We also discuss choice environments where the multiplicative inequality is implausible.

Keywords: discrete choice, multiplicative inequality, regularity, EBA model

1 Introduction

Most theoretical models of choice in economics are built upon algebraic preferences, rather than stochastic foundations. At the same time, human choice seems irreducibly uncertain. Most applied work reconciles deterministic models with stochastic data by positing either random preferences or error terms. Usually, distributions of preferences and errors are given without theoretical justification. Part of Duncan Luce's challenge (Luce (1995), Luce (1997)) is to build choice models that generalize existing deterministic models and that are supported by stochastic foundations. In a survey of stochastic utility, Fishburn (1999) documents many axioms and conditions on choice probabilities that have been studied in the literature. Many of these pertain to binary choice probabilities, natural analogues of binary algebraic preferences, including nine stochastic analogues of transitivity.

Most of the choices economists study are from sets with more than two options. Once algebraic preferences are specified, and constrained by conditions such as transitivity and others, there is usually little more to add with regard to multiple choice, since in any choice set, a unique option is preferred to all others. Modelling stochastic choice from sets with more than two options is

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much less straightforward and requires structure beyond that provided by axioms on binary choice probabilities alone.

A few axioms have been proposed that relate choice probabilities across choice sets of different sizes. One of them is the multiplicative inequality (MI) introduced by Sattath and Tversky (1976), who also establish that the MI is a necessary condition for two different choice models: Tversky's (1972b) Elimination by Aspects (EBA) model, and the independent random utility model. The MI, defined below, is simple, intuitively appealing, and testable.

Our main contribution, Theorem 3.1, shows how little the MI constrains binary choice probabilities, revealing a complementarity between the MI and any axioms for binary choice probabilities that one might wish to impose. The theorem implies that any system of binary choice probabilities on a universe of objects can be extended to a complete system of choice probabilities satisfying the MI. But is stronger than that. Consider a system of binary probabilities on one universe and the extension of this to a system of binary probabilities on a larger universe containing the first. The theorem implies that no matter how we extend the system of binary probabilities on the smaller universe to a complete system of probabilities on that universe satisfying the MI, we can then extend the system of binary probabilities on the larger universe to a complete system satisfying the MI in such a way that the latter complete system is the extension of the former to the larger universe. In this way, the extension result in Theorem 3.1 is more flexible than a recently published result on the extension of binary choice probabilities satisfying the triangle inequality to systems of probabilities satisfying regularity, another simple, intuitively appealing and testable condition relating choice probabilities across choice sets of different sizes.

Another contribution is a series of illustrations, theorems and counterexamples shedding light on the relationship between regularity and the MI. We illustrate graphically the complementarity between them. For systems of binary and ternary choice probabilities, Theorem 2.1 establishes a kind of compatibility between regularity and the MI, while Theorem 2.2 shows the equivalence between the existence of an independent random utility representation of binary choice probabilities and the condition that the set of extensions satisfying regularity is *not* included in the interior of the set of extensions satisfying the MI. Theorem 3.3 shows that while the set of choice probabilities satisfying regularity is convex, the set satisfying the MI is not. Theorem 3.4 shows that with regularity maintained, many instances of the multiplicative inequality are redundant.

A final contribution is a cautionary observation, suggesting that there are choice environments where the multiplicative inequality is implausible.

1.1 Preliminaries

Let $T = \{x_1, \ldots, x_n\}$ be a universe of choice objects. When faced with a non-empty choice set $A \subseteq T$, an agent chooses a single object from A. The probability that the agent chooses $x \in A$ is denoted $P_A(x)$. A *system of choice probabilities* P on T is the complete specification of the $P_A(x)$, $x \in A \subseteq T$. For distinct $x, y \in T$, we use the shorthand notation p_{xy} to mean $P_{\{x,y\}}(x)$. A *binary system of choice probabilities* p on T is the complete specification of p_{xy} for all distinct $x, y \in T$. We will abbreviate these two terms and speak of a *system* P or a *binary system* p. We define δ_T as the set of binary systems on T and Δ_T as the set of systems on T; they are both Cartesian products of simplices. For a given binary system $p \in \delta_T$, we define $\Delta_T(p)$ as the set of systems on T that extend p; that is, the set of systems on T whose binary probabilities coincide with p.

In this paper, we will be particularly interested in the two following conditions and the relationship between them. The system of choice probabilities P on the universe T satisfies the *multiplicative inequality* if for all $x \in A, B \subseteq T$,

$$P_{A \cup B}(x) \geq P_A(x) P_B(x), \tag{1}$$

and *regularity* if for all $x \in A \subseteq B \subseteq T$,

$$P_A(x) \geq P_B(x); \tag{2}$$

We will also be interested in the relationship between binary choice probabilities and multiple choice probabilities implied by these axioms, and so the following definitions will be useful. For a universe T, we define M_T as the subset of Δ_T where the multiplicative inequality holds and R_T as the subset where regularity holds. Just as $\Delta_T(p)$, the set of systems extending the binary system p, is defined as a cross section of Δ_T, we can define the following cross sections of M_T and R_T. For each $p \in \delta_T$, let $M_T(p) = \Delta_T(p) \bigcap M_T$, the set of systems extending the binary system p and satisfying the MI, and $R_T(p) = \Delta_T(p) \bigcap R_T$, the set of systems extending p and satisfying regularity.

1.2 Conditions relating multiple choice probabilities

Here we describe some of the conditions that have been proposed to constrain choice probabilities across choice sets of different sizes. Of particular importance here is the multiplicative inequality, the subject of the paper, and regularity, whose connection with the MI is particularly important, but we will also consider some well known alternative conditions in the literature. We will compare all these conditions on the basis of interpretability, testability, empirical evidence, and any constraints over binary choice probabilities that they imply. Conditions on choice probabilities mentioned in the paper, other than regularity and the MI defined above, are defined in Appendix A.

Luce's (1959) choice axiom relates choice probabilities across choice sets in a particularly restrictive way. When binary choice probabilities are non-degenerate, choice probabilities satisfy a condition known as the *constant ratio rule* or Independence from Irrelevant Alternatives (IIA). According to this rule, not only does a system of binary choice probabilities uniquely determine all other choice probabilities, the binary choice probabilities themselves are tightly constrained: for all distinct $x, y, z \in T$, $p_{xz}p_{zy} = p_{xy}$ must hold. The choice axiom has been criticized on both intuitive and empirical grounds. As much of the criticism also applies to the weaker simple scalability condition, we refer to it below.

According to Krantz's (1964) *simple scalability* condition, choice probabilities are functions only of the feasible objects' univariate and context free "scale" values, and these functions adhere to certain intuitive monotonicity conditions. Tversky (1972b) shows that simple scalability is equivalent to the *order independence* condition, which is expressed in terms of inequalities over choice probabilities and thus directly testable. Tversky (1972b) points out that order independence can be seen as a weaker, ordinal, version of the constant ratio rule, but it too has strong implications for choice probabilities and has been criticized on empirical grounds. Tversky (1972a) summarizes the theoretical problems with and empirical evidence against simple scalability. Rieskamp, Busemeyer, and Mellers (2006) show that simple scalability implies strong stochastic transitivity (SST) of binary choice probabilities and reference some of the empirical evidence against SST.

Regularity, the condition that adding objects to a choice set cannot increase the probability of choosing an object already in the choice set, is simple and intuitively appealing. Dasgupta and Pattanaik (2007) make the case that "given its weakness, transparency, and intuitive appeal, regularity recommends itself as the natural axiomatic foundation for the theory of choice behavior". Regularity has largely held up to empirical scrutiny, but the asymmetric dominance effect documented by Huber, Payne, and Puto (1982) and others is inconsistent with it. Luce and Suppes (1965) show that regularity implies the triangle inequality for binary choice probabilities and Sprumont (2022) shows that any binary system satisfying the triangle inequality can be extended to a system satisfying regularity.

Falmagne (1978) showed that the set of Block-Marschak conditions (Block and Marschak (1960)) is equivalent to random utility. This makes it possible to test random utility, but while random utility is easy to understand, the Block-Marschak conditions are neither simple nor intuitively appealing in themselves. As regularity is a necessary condition for random utility—see Luce and Suppes (1965)—random utility is also inconsistent with the asymmetric dominance effect and also implies the triangle inequality for binary choice probabilities. McCausland and Marley (2014) and McCausland, Davis-Stober, Marley, Park, and Brown (2020) describe direct tests of the complete set of Block-Marschak conditions, and apply them to individual choice data, finding choice behaviour consistent with these conditions for all but a few individuals.

Sattath and Tversky (1976) introduce the multiplicative inequality (MI) and claim "it has some appeal as a principle of individual choice behaviour." One interpretation of the MI is that the probability of choosing x from $A \cup B$ is at least as great as the probability of choosing x from A and again from B in independent trials. In the special case where a system P is induced by a strict random preference $\succ$, $P_A(x)$, $P_B(x)$ and $P_{A\cup B}(x)$ are the probabilities of the events $\Lambda_A(x) \equiv \{x \succ y \colon y \in A, y \neq x\}$, $\Lambda_B(x) \equiv \{x \succ y \colon y \in B, y \neq x\}$ and $\Lambda_{A\cup B}(x) \equiv \{x \succ y \colon y \in A \cup B, y \neq x\} = \Lambda_A(x) \cap \Lambda_B(x)$. In this case, another interpretation of the MI is that it requires that the events $\Lambda_A(x)$ and $\Lambda_B(x)$ be positively associated; equivalently, that learning that x is the most preferred element of A increases the probability that it is also the most preferred element of B. Colonius (1983) makes a related point about the MI when P is induced by random utility.

Sattath and Tversky (1976) also point out the following interesting complementarity between regularity and the MI. Suppose both conditions hold for a universe T. Then for all $A, B \subseteq T$ and all $x \in A \cap B$,

$$\min(P_A(x), P_B(x)) \geq P_{A\cup B}(x) \geq P_A(x)P_B(x). \tag{3}$$

Regularity gives the upper bound for $P_{A\cup B}(x)$, in terms of $P_A(x)$ and $P_B(x)$, and the MI gives the lower bound.

The literature on MI is sparse compared with that on EBA. Corbin and Marley (1974) cast some doubt on the MI with a plausible hypothetical description of a researcher considering three candidates for a position. The researcher chooses whether the position is for a graduate student or for a research assistant; both are equally likely. The universe of candidates is $T = \{x, y, z\}$ and they are ranked $x \succ y \succ z$ in terms of their suitability as a graduate student and $z \succ y \succ x$ as a potential research assistant. This random preference induces choice probabilities that violate the MI: $P_T(y) = 0 < \frac{1}{4} = p_{yx}p_{yz}$. McCausland and Marley (2014) and McCausland, Davis-Stober, Marley, Park, and Brown (2020) perform tests of the complete set of MI conditions, finding much individual variation in evidence for or against the MI.

1.3 Models satisfying the multiplicative inequality

In a doubly seminal paper, Tversky (1972b) introduces both the similarity effect, and the Elimination by Aspects (EBA) model. The similarity effect describes a pattern of stochastic choice behaviour that is well documented but inconsistent with the constant ratio rule. Some versions of the similarity effect (see Davis-Stober, Marley, McCausland, and Turner (2023) for details) are inconsistent with simple scalability, a property of most discrete choice models. The Elimination by Aspects (EBA) model describes choice as arising from a latent stochastic sequential process of elimination. It does not imply simple scalability, and for suitable choices of the model's parameters, the resulting choice probabilities are consistent with the similarity effect.

The EBA model has other advantages. McFadden (1981) affirms that it "can accommodate complex patterns of substitutability of alternatives" and suggests that it has "considerable potential for econometric applications." A result not unrelated to McFadden's first comment is that EBA allows for choice probabilities that are ruled out by independent random utility; Tversky (1972a) (page 346) provides an example of plausible choice probabilities that can be generated by EBA but not by any independent random utility model.

Several testable implications of EBA have been identified. Tversky (1972a) shows that EBA models satisfy regularity (Theorem 3 in that paper) and moderate stochastic transitivity (Theorem 4). Sattath and Tversky (1976) show that EBA models satisfy the MI. Tversky (1972a, Theorem 7) shows that EBA is a random utility model.

Sattath and Tversky (1976) also show that independent random utility models, where utilities are statistically independent, imply the MI. That said, by far the most commonly used independent random utility model is the multinomial logit model, which is already covered as a special case of EBA.

1.4 Outline

In Section 2, we consider the important special case where the universe T has three elements. We prove some new results on the multiplicative inequality and its relationship with regularity and independent random utility, accompanied by graphical illustrations. In Section 3, we prove some results for general T. In Section 4, we describe plausible random preferences where the MI is violated. In Section 5, we conclude.

2 Results and illustrations for a universe with three elements

In this section, we consider the special case where $|T| = 3$. This is an important case as many so-called *context effects* are framed as conditions relating choice probabilities on tripleton choice sets and doubleton subsets. Davis-Stober, Marley, McCausland, and Turner (2023) survey the literature on three context effects well known in the economics, marketing and mathematical psychology literature, including the similarity and asymmetric dominance effects mentioned above and the compromise effect introduced by Simonson (1989) and Tversky and Simonson (1993). Davis-Stober, Marley, McCausland, and Turner (2023) present previously known and new results relating different versions of these effects and various conditions on systems, including regularity, random utility, the constant ratio rule and simple scalability, all mentioned above. The case $|T| = 3$ is also one where graphics can be used to illustrate the cross sections $R_T(p)$ and $M_T(p)$ of the regularity and MI regions. We begin by explaining a graphical device we use to illustrate choice probabilities on tripleton sets and their doubleton subsets, as well as to illustrate the cross sections $M_T(p)$ and $R_T(p)$ for various binary systems p.

2.1 Binary and ternary choice probabilities in a Barycentric coordinate system

Let $T = \{x, y, z\}$. We first describe a way of plotting all binary and ternary choice probabilities associated with T as points in a single Barycentric coordinate system, as illustrated in Figure 1. Each point on the equilateral triangle xyz, including the interior and the boundary, represents a triple $(P_T(x), P_T(y), P_T(z))$ of choice probabilities. The vertices x, y and z represent the probability vectors $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$, meaning the certain choice of objects x, y and z, respectively. In general, the probability vector $(P_T(x), P_T(y), P_T(z))$ is the convex combination of the vertices x, y and z, with weights $P_T(x)$, $P_T(y)$ and $P_T(z)$. For example, the interior point a is the probability vector $(P_T(x), P_T(y), P_T(z)) = (\frac{1}{2}, \frac{3}{10}, \frac{1}{5})$. The fact that xyz is equilateral means that if we normalize the height of the triangle to one, the distance of a point $(P_T(x), P_T(y), P_T(z))$ to the right side of the triangle is $P_T(x)$; to the base, $P_T(y)$; and to the left side, $P_T(z)$.

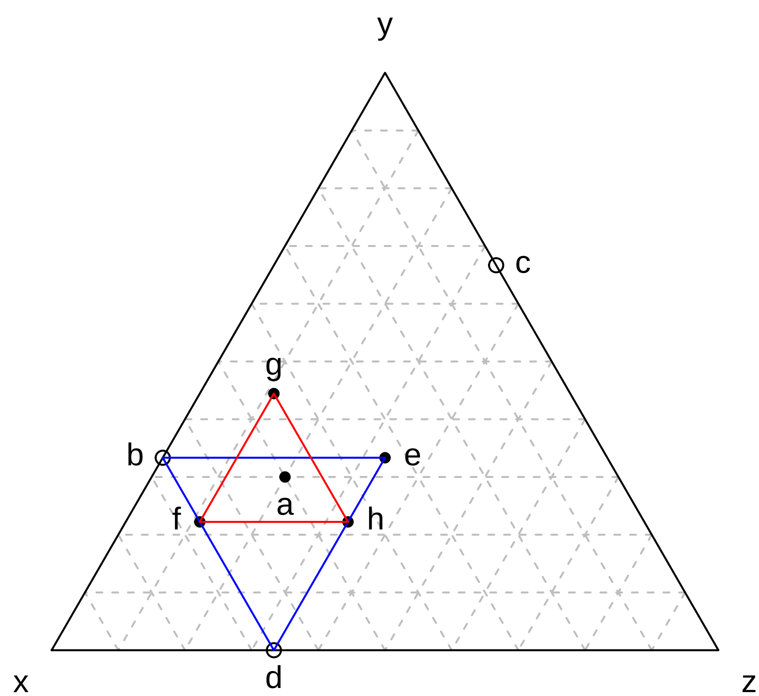

Figure 1: Binary and ternary probabilities on a universe $T = \{x, y, z\}$ in a Barycentric coordinate system. Points a, b, c and d specify a system on T satisfying both regularity and the MI; b, c and d alone specify a binary system on T. Solid triangle bde gives the set of ternary probabilities (which includes a) extending the binary system to a system satisfying regularity. Solid triangel fgh gives the set of ternary probabilities (which also includes a) extending the binary system to a system satifying the MI.

We also represent the binary probabilities p_{xy}, p_{yz} and p_{xz} as points on the left, right and bottom sides of the triangle, respectively. We use hollow dots for binary probabilities and solid dots for ternary probabilities, to avoid ambiguity for points on the sides of the triangle. A point on the left side of the triangle is the convex combination of the vertices x and y, with weights p_{xy} and p_{yx} respectively; similarly, a point on the right side is the convex combination of y and z with weights p_{yz} and p_{zy}, and a point on the base is the convex combination of z and x with weights p_{zx} and p_{xz}. In Figure 1, the points b, c and d represent the binary probabilities $p_{xy} = \frac{2}{3}$, $p_{yz} = \frac{2}{3}$ and $p_{zx} = \frac{1}{3}$.

2.2 Regularity and the MI

For a fixed binary system p on $T = \{x, y, z\}$, we now consider the cross sections $R(p)$ and $M(p)$, the sets of extensions of p to systems satisfying regularity and the MI, respectively. By definition,

$$R_T(p) \equiv \{P \in \Delta_T(p) \colon P_T(x) \leq \min(p_{xy}, p_{xz}),\ P_T(y) \leq \min(p_{yx}, p_{yz}),\ P_T(z) \leq \min(p_{zx}, p_{zy})\} \quad (4)$$

and

$$M_T(p) \equiv \{P \in \Delta_T(p) \colon P_T(x) \geq p_{xy}p_{xz},\ P_T(y) \geq p_{yx}p_{yz},\ P_T(z) \geq p_{zx}p_{zy}\}. \quad (5)$$

Take as an example the binary system p given by the points b, c and d; that is, where $p_{xy} = \frac{2}{3}$, $p_{yz} = \frac{2}{3}$ and $p_{zx} = \frac{1}{3}$. This binary system is extended to a system on T by specfying the ternary choice probability vector $(P_T(x), P_T(y), P_T(z))$. The extension satisfies the MI if and only if $(P_T(x), P_T(y), P_T(z)) \geq (\frac{4}{9}, \frac{2}{9}, \frac{1}{9})$; the region of compatible ternary choice probabilities is the equilateral triangle fgh in Figure 1. The extension satisfies regularity if and only if $(P_T(x), P_T(y), P_T(z)) \leq (\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$; here the region is the equilateral triangle bde. It is clear from equations (4) and (5) that if they exist, the set of ternary choice probabilities extending any binary system on T to a system satisfying the MI will always be an equilateral triangle pointing up and the set of ternary choice probabilities extending a binary system on T to a system satisfying regularity will always be an equilateral triangle pointing down, with the convention that a single point is a special case of both.

Clearly, regularity and the MI are logically independent. For the binary system p given above, defined by $p_{xy} = \frac{2}{3}$, $p_{yz} = \frac{2}{3}$ and $p_{zx} = \frac{1}{3}$, it is easy to check directly that the extension of p with $(P_T(x), P_T(y), P_T(z)) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ (point e) satisfies regularity and not the multiplicative inequality. The extension with $(P_T(x), P_T(y), P_T(z)) = (\frac{4}{9}, \frac{4}{9}, \frac{1}{9})$ (point g) satisfies the multiplicative inequality and not regularity. The extension with $(P_T(x), P_T(y), P_T(z)) = (0, 0, 1)$ (point z) satisfies neither and the extension with $(\frac{1}{2}, \frac{3}{10}, \frac{1}{5})$ (point a) satisfies both.

As an aside, the MI is in fact logically independent of many well-known axioms of discrete probabilistic choice. Section 2.2 of McCausland and Marley (2013) gives a complete description, with a diagram, of the logical relationships among the following conditions: weak, moderate and strong stochastic transitivity; the triangle inequality; regularity; the Block-Marschak conditions; and the MI. See Appendix A for definitions of conditions that do not appear in the main text. Not only is the MI logically independent of each of the other conditions in this list, any logically possible combination of the other conditions being satisfied or not is compatible both with the MI being satisfied and with it not being satisfied.

For some choices of p_{xy}, p_{yz} and p_{zx}, $R(p)$ is empty. For example, if $p_{xy} = p_{yz} = p_{zx} = \frac{1}{4}$, regularity requires $P_T(x) \leq \frac{1}{4}$, $P_T(y) \leq \frac{1}{4}$ and $P_T(z) \leq \frac{1}{4}$, which is impossible. Sprumont (2022) shows for general T that $R(p)$ is non-empty if and only if binary system p satisfies the triangle inequality; for $T = \{x, y, z\}$ under discussion here, this condition can be summarized as $1 \leq p_{xy} + p_{yz} + p_{zx} \leq 2$.

In contrast, while $M(p)$ can be a single point, it is never empty. This, too, is true for general T, by Corollary 3.2 below, but it will be instructive to show this for $T = \{x, y, z\}$ using a simpler approach. Consider the equation

$$\begin{aligned} 1 &= (p_{xy} + p_{yx})(p_{xz} + p_{zx})(p_{yz} + p_{zy}) \\ &= p_{xy}p_{xz} + p_{yx}p_{yz} + p_{zx}p_{zy} + p_{xy}p_{yz}p_{zx} + p_{xz}p_{zy}p_{yx}. \end{aligned}$$

The first three terms on the second line give the lower bounds on $P_T(x)$, $P_T(y)$ and $P_T(z)$, respectively, implied by the MI. The remainder $S(p) \equiv p_{xy}p_{yz}p_{zx} + p_{xz}p_{zy}p_{yx}$ clearly must satisfy $0 \leq S(p) \leq 1$. $S(p)$ measures the total amount of slack in the three multiplicative inequalities $P_T(x) \geq p_{xy}p_{xz}$, $P_T(y) \geq p_{yx}p_{yz}$ and $P_T(z) \geq p_{zx}p_{zy}$, and so the height of the equilateral triangle of ternary probabilities consistent with the MI for given p is a fraction $S(p)$ of the height of the triangle xyz.

Clearly, $S(p) = 0$ if and only if there is at least one object that is either always chosen or never chosen in pairwise comparisons with the other two objects. In this case, the ternary probability of such an object is constrained to be one or zero, respectively, and the set of ternary probability vectors consistent with the MI and given p is the single point $(p_{xy}p_{xz}, p_{yz}p_{yx}, p_{zx}p_{zy})$. Also, $S(p) = 1$ if and only if there is a deterministic intransitive cycle in one direction ($p_{xy} = p_{yz} = p_{zx} = 1$) or the other ($p_{xy} = p_{yz} = p_{zx} = 0$), in which case the ternary probability vector is completely unconstrained by the MI.

Figure 2 shows ten binary-ternary plots, each for a different binary system p. Each figure shows the set of ternary probability vectors $P_T(\cdot)$ extending p to a system in $M_T(p)$ and the set of ternary probability vectors extending p to a system in $R_T(p)$. Panels (a), (b) and (c) show the cases $p_{xy} = p_{yz} = p_{zx} = 0.9$, $p_{xy} = p_{yz} = p_{zx} = 0.65$ and $p_{xy} = p_{yz} = p_{zx} = 0.5$, respectively. In (a), weak stochastic transitivity (WST) is violated and $R_T(p)$ is empty. In (b), WST is violated but $R_T(p)$ is not empty. In (c), the binary choice probabilities are on the boundary of the WST region; relative to (b), $M(p)$ changes little but $R(p)$ is much larger. In panels (d) and (e), the stochastically intransitive cycles in (b) and (a), respectively, are broken by replacing p_{zx} with its complementary probability; the regularity and MI cross sections change considerably. Binary probabilities in (g) show x and y usually chosen over z in pairwise comparison and a kind of indifference between x and y; those in (h) show x usually chosen over y and z in pairwise comparison and a kind of indifference between y and z. Binary probabilities in (i) show x usually chosen over y but indifference between x and z and between y and z; this pattern is more unexpected, and like the cases with stochastically intransitive cycles, $M(p)$ is particularly large relative to $R(p)$. Panel (j) illustrates a tendency for both $R_T(p)$ and $M_T(p)$ to grow larger when binary choice probabilities become more moderate.

We have seen in equation (3) a kind of complementary between regularity and the MI. However, adding-up constraints mean that they are, in one sense, similar. The upper bounds on $P_T(x)$, $P_T(y)$ and $P_T(z)$ given by regularity, together with the condition that $P_T(x) + P_T(y) + P_T(z) = 1$, imply lower bounds as well: $P_T(x) \geq 1 - \min(p_{yz}, p_{yx}) - \min(p_{zx}, p_{zy})$ and similar expressions for $P_T(y)$ and $P_T(z)$. These lower bounds are often similar to the lower bounds given by the MI. Likewise, the lower bounds on $P_T(x)$, $P_T(y)$ and $P_T(z)$ given by the MI, together with $P_T(x)+P_T(y)+P_T(z) = 1$, imply upper bounds, which are often close to the upper bounds given by regularity. For many binary systems p, the $M(p)$ and $R(p)$ regions are in similar positions and have similar sizes, especially for more plausible binary systems, making empirical discrimination between regularity and MI difficult. Looking at all ten panels, we see that for the binary system p of each panel, $M(p)$ and $R(p)$ intersect whenever $R(p)$ is not empty. Although panel (e) shows that it is possible for $M(p)$ and $R(p)$ to be nearly disjoint, in fact they intersect for all $p \in \delta_T$, $|T| = 3$, as the following theorem implies.

Theorem 2.1 *For all* $p \in \delta_T$, $|T| = 3$, $R(p) \neq \emptyset \Rightarrow R(p) \cap M(p) \neq \emptyset$.

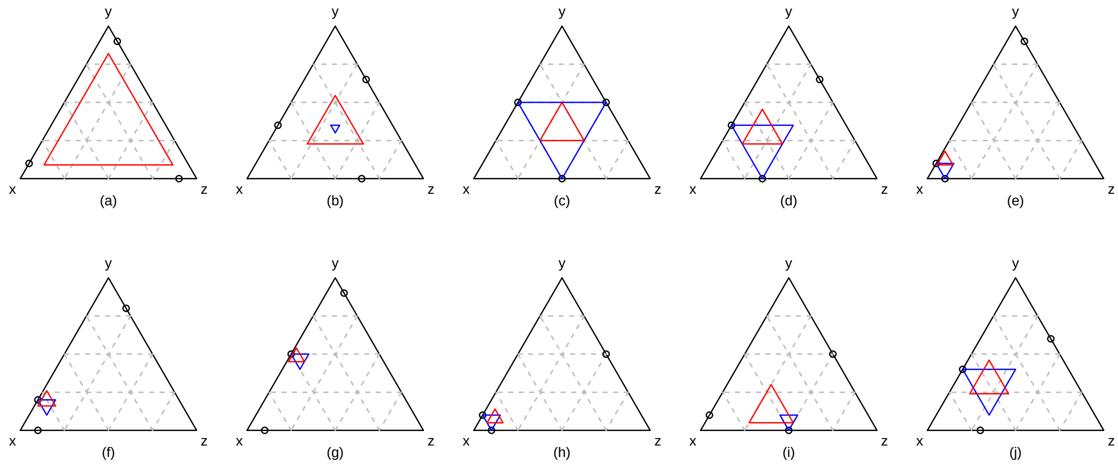

Figure 2: A Gallery of binary choice probabilities and regions where the ternary probabilities satisfy regularity (downward pointing blue triangle and its interior) and the multiplicative inequality (upward pointing red triangle and its interior).

Proof Let $T = \{x, y, z\}$ and let $p \in \delta_T$ be arbitrary. Let $a = p_{xy}p_{xz} + p_{yz}p_{yx} + p_{zx}p_{zy}$ and $b = \min(p_{xy}, p_{xz}) + \min(p_{yz}, p_{yx}) + \min(p_{zx}, p_{zy})$. Since $a = 1 - S(p)$, $0 \leq a \leq 1$. Now suppose that $R(p) \neq \emptyset$, which implies $b \geq 1$. Together, we have $a \leq 1 \leq b$.

Consider first the case $a = b$. Then since $p_{xy}p_{xz} \leq \min(p_{xy}, p_{xz})$, $p_{yz}p_{yx} \leq \min(p_{yz}, p_{yx})$ and $p_{zx}p_{zy} \leq \min(p_{zx}, p_{zy})$, it must be that $p_{xy}p_{xz} = \min(p_{xy}, p_{xz})$, $p_{yz}p_{yx} = \min(p_{yz}, p_{yx})$ and $p_{zx}p_{zy} = \min(p_{zx}, p_{zy})$. Now let $P_T(x) = p_{xy}p_{xz}$, $P_T(y) = p_{yz}p_{yx}$ and $P_T(z) = p_{zx}p_{zy}$. Regularity and the MI are both satisfied with equality and $P_T(x) + P_T(y) + P_T(z) = a = b = 1$.

Now consider the case $a < b$. Let

$$\alpha \equiv \frac{b-1}{b-a}.$$

Since $a \leq 1 \leq b$, $0 \leq \alpha \leq 1$. Now construct

$$P_T(x) = \alpha p_{xy}p_{xz} + (1-\alpha)\min(p_{xy}, p_{xz}),$$

$$P_T(y) = \alpha p_{yz}p_{yx} + (1-\alpha)\min(p_{yz}, p_{yx}),$$

$$P_T(z) = \alpha p_{zx}p_{zy} + (1-\alpha)\min(p_{zx}, p_{zy}).$$

Since $0 \leq \alpha \leq 1$, $P_T(x)$, $P_T(y)$ and $P_T(z)$ are all non-negative. Also

$$P_T(x) + P_T(y) + P_T(z) = \alpha a + (1-\alpha)b = 1.$$

Since each of $P_T(x)$, $P_T(y)$ and $P_T(z)$ is a convex combination of the lower bound imposed by the MI and the upper bound imposed by regularity, $P \in R(p) \cap M(p)$. □

We do not know whether Theorem 2.1 extends to $|T| > 3$. In Section 5, we discuss the prospects of a constructive proof along the lines of the proof of Theorem 3.1, and identify some difficulties standing in the way.

We notice that $M(p)$ tends to be large when the the binary probabilities are far from satisfying the triangle inequality, a condition that is necessary for regularity and described by Fishburn (1999) as one of many kinds of stochastic transitivity. We can make this observation more precise by considering the minimization and maximization of $S(p)$ with respect to the vector $\bar{p} = (p_{xy}, p_{yz}, p_{zx})$. The gradient and Hessian of $S(p)$ with respect to $\bar{p}$ are

$$\frac{dS(p)}{d\bar{p}} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_{xy} \\ p_{yz} \\ p_{zx} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad \frac{d^2 S}{d\bar{p} d\bar{p}^\top} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The function $S(p)$ has a unique critical point at $p_{xy} = p_{yz} = p_{zx} = \frac{1}{2}$ and the eigenvalues of the Hessian are -1, -1 and 2 everywhere. Thus, the critical point is a saddle point, and so $S(p)$ cannot have local minima or maxima in the interior of $[0,1]^3$. Consider now the minimization and maximization of $S(p)$ as a function of p_{xy} for fixed p_{yz} and p_{zx}. First note that

$$p_{yz}p_{zx} > p_{xz}p_{zy} \Leftrightarrow p_{yz} + p_{zx} > 1 \Leftrightarrow p_{xz} + p_{zy} < 1.$$

If $p_{yz}p_{zx} > p_{xz}p_{zy}$, $S(p)$ is minimized at $p_{xy} = 0$, where the triangle inequality holds ($p_{xy} + p_{yz} \geq p_{xz}$ and $p_{xz} + p_{zy} \geq p_{xy}$) and maximized at $p_{xy} = 1$, where the triangle inequality is violated ($p_{xz} + p_{zy} < p_{xy}$). Similarly, if $p_{yz}p_{zx} < p_{xz}p_{zy}$, $S(p)$ is minimized at $p_{xy} = 1$, where the triangle inequality holds, and maximized at $p_{xy} = 0$, where the triangle inequality is violated. In both cases, $S(p)$ monotonically increases as p_{xy} goes from the value where the triangle inequality is satisfied with the most slack to the value where it is violated with the most slack.

We see that much of the volume of M_T features unrealistic choice probabilities, in the form of violations of the triangle inequality or regularity. This observation is also supported by an analysis in McCausland and Marley (2013) showing that for a wide range of prior distribution over Δ_T (for $|T| = 3, 4, 5$), the prior probability of the MI is much greater than the probability of regularity. This has implications for Bayesian inference and particularly the interpretation of Bayes factors in favour of systems restricted to the MI region versus systems that are not. Bayes factors will favour an unconstrained encompassing model over a restriction of that model to M_T, relative to what one might expect if unaware of the large prior probability of unrealistic systems satisfying the MI. One recommendation that emerges is that rather than testing the MI against a completely unconstrained model, one should test the MI jointly in conjunction with other stochastic choice axioms, or test the MI within an encompassing model constrained by some maintained axioms. In both cases, these axioms should be ones that rule out the most unrealistic parts of M_T. Results and observations in this paper suggest some suitable candidates: regularity and various axioms for binary choice probabilities. We have seen ways in which regularity and the MI are complementary and Theorem 2.1 shows some compatability between them. Theorem 3.1 below implies that any binary system can be extended to a system satisfying the MI, revealing a strong type of compatibility between the MI and *any* axiom for binary choice probabilities.

2.3 The multiplicative inequality and independent random utility

Suck (2002) considers the question of which binary systems admit an independent random utility representation. For a universe T of arbitrary size, his Theorem 2 gives a necessary condition: for all distinct $x, y, z \in T$, $\min(p_{xy} + p_{yz}p_{zx}, p_{yz} + p_{zx}p_{xy}, p_{zx} + p_{xy}p_{yz}) \leq 1$. For $|T| = 3$, the condition is also sufficient, and his proof of this is constructive.

The following theorem establishes a simple geometric interpretation for Suck's (2002) necessary and sufficient conditions for the case $|T| = 3$:

Theorem 2.2 *For all $p \in \delta_T$, $|T| = 3$, p admits an independent random utility representation if and only if $R(p) \not\subseteq \text{interior}(M(p))$.*

The result is mildly paradoxical: regularity and the MI are both necessary conditions for an independent random utility representation, but if *all* the regular extensions of a binary system satisfy the MI with strict inequalities, there *cannot* be such a representation. In Figure 2, panels (a) and (b) show binary systems inconsistent with independent random utility. Only in panel (b), where $R(p)$ is non-empty, are there completions of the binary system p satisfying random utility but none satisfying independent random utility.

Proof Let $T = \{x, y, z\}$ and $p \in \delta_T$ be arbitrary. First consider the case $R_T(p) = \emptyset$. Since regularity is a necessary condition for a random utility representation, p does not admit a random utility representation, let alone an independent random utility representation. At the same time $R(p) = \emptyset \subseteq \text{interior}(M(p))$ trivially.

Now consider the case $R_T(p) \neq \emptyset$. Then the maximum allowable values of $P_T(x)$, $P_T(y)$ and $P_T(z)$ must have a sum greater than or equal to one. That is,

$$\min(p_{xy}, p_{xz}) + \min(p_{yx}, p_{yz}) + \min(p_{zx}, p_{zy}) \geq 1. \tag{6}$$

Now consider the following three conditions.

1. All of the following hold:
   (a) $p_{xy} + p_{yz}p_{zx} > 1$ (equivalent to $p_{xy} - p_{xz} > p_{zx}p_{zy}$),
   (b) $p_{yz} + p_{zx}p_{xy} > 1$ (equivalent to $p_{yz} - p_{yx} > p_{xy}p_{xz}$),
   (c) $p_{zx} + p_{xy}p_{yz} > 1$ (equivalent to $p_{zx} - p_{zy} > p_{yz}p_{yx}$).
2. All of the following hold:
   (a) $p_{yx} + p_{xz}p_{zy} > 1$ (equivalent to $p_{yx} - p_{yz} > p_{zx}p_{zy}$),
   (b) $p_{xz} + p_{zy}p_{yx} > 1$ (equivalent to $p_{xz} - p_{xy} > p_{yz}p_{yx}$),
   (c) $p_{zy} + p_{yx}p_{xz} > 1$ (equivalent to $p_{zy} - p_{zx} > p_{xy}p_{xz}$).
3. All of the following hold:
   (a) $1 - \min(p_{xy}, p_{xz}) - \min(p_{yx}, p_{yz}) > p_{zx}p_{zy}$,
   (b) $1 - \min(p_{yx}, p_{yz}) - \min(p_{zx}, p_{zy}) > p_{xy}p_{xz}$,
   (c) $1 - \min(p_{xy}, p_{xz}) - \min(p_{zx}, p_{zy}) > p_{yx}p_{yz}$.

By corollary 4 in Suck (2002), p does not admit an independent random utility representation if and only if either of conditions 1 or 2 hold. The equivalence of the conditions in parentheses is easy to verify and they are quite useful in the proof.

We now show that for $R_T(p) \neq \emptyset$, $R_T(p) \subseteq \text{interior}(M_T(p))$ is equivalent to condition 3 above. The set of ternary probability vectors $(P_T(x), P_T(y), P_T(z))$ extending p to a system consistent with regularity—see (4)—is an equilateral triangle, the set of convex combinations of the vertices

$$(\min(p_{xy}, p_{xz}), \min(p_{yx}, p_{yz}), 1 - \min(p_{xy}, p_{xz}) - \min(p_{yx}, p_{yz})),$$

$$(1 - \min(p_{yx}, p_{yz}) - \min(p_{zx}, p_{zy}), \min(p_{yx}, p_{yz}), \min(p_{zx}, p_{zy})),$$

and

$$(\min(p_{xy}, p_{xz}), 1 - \min(p_{xy}, p_{xz}) - \min(p_{zx}, p_{zy}), \min(p_{zx}, p_{zy})).$$

Suppose $R_T(p) \subseteq \text{interior}(M_T(p))$. Then all of these vertices must strictly dominate the vector $(p_{xy}p_{zy}, p_{yx}p_{yz}, p_{zx}p_{zy})$, which gives condition 3.

Now suppose that Condition 3 holds. Together with equation (6), condition 3 implies $\min(p_{xy}, p_{xz}) > p_{xy}p_{xz}$, $\min(p_{yx}, p_{yz}) > p_{yx}p_{yz}$, $\min(p_{zx}, p_{zy}) > p_{zx}p_{zy}$ and that each of the vertices above extends p to a system consistent with the MI. Since the set of ternary probability vectors $(P_T(x), P_T(y), P_T(z))$ extending p to a system consistent with regularity is the set of convex combinations of these vertices, and since the set of ternary probability vectors $(P_T(x), P_T(y), P_T(z))$ extending p to a system consistent with the MI is convex, $R_T(p) \subseteq \text{interior}(M_T(p))$.

We now show that condition 3 holds if and only if either condition 1 or condition 2 holds. (Note in passing that at most one of conditions 1 and 2 can hold; this is most obvious looking at the conditions in parentheses.) Suppose that condition 1 holds. Using the versions of the conditions 1a, 1b and 1c in parentheses, first to establish which argument of each $\min(\cdot,\cdot)$ function is minimal and then again in the final inequality, we obtain

$$1 - \min(p_{xy}, p_{xz}) - \min(p_{yx}, p_{yz}) = 1 - p_{xz} - p_{yx} = p_{xy} - p_{xz} > p_{zx}p_{zy},$$
$$1 - \min(p_{yx}, p_{yz}) - \min(p_{zx}, p_{zy}) = 1 - p_{yx} - p_{zy} = p_{yz} - p_{yx} > p_{xy}p_{xz},$$
$$1 - \min(p_{xy}, p_{xz}) - \min(p_{zx}, p_{zy}) = 1 - p_{zy} - p_{xz} = p_{zx} - p_{zy} > p_{yz}p_{yx}.$$

This establishes condition 3. The proof that condition 2 implies condition 3 is very similar.

We now complete the proof by showing that condition 3 implies that either of conditions 1 and 2 hold; we have already seen that they cannot both hold. Suppose that condition 3 holds and consider first the condition 3a. We can rule out the case where $\min(p_{xy}, p_{xz}) = p_{xy}$ and $\min(p_{yx}, p_{yz}) = p_{yx}$ because this makes the left hand side of 3a equal to 0, which cannot be greater than the right hand side. We can also rule out the case where $\min(p_{xy}, p_{xz}) = p_{xz}$ and $\min(p_{yx}, p_{yz}) = p_{yz}$, as this makes the left hand side of 3a equal to $1 - p_{xz} - p_{yz}$ and

$$1 - p_{xz} - p_{yz} \leq (1 - p_{xz} - p_{yz} + p_{xz}p_{yz}) = p_{zx}p_{zy},$$

a direct contradiction of 3a. This leaves two possibilities:

$$\min(p_{xy}, p_{xz}) = p_{xz} \quad \text{and} \quad \min(p_{yx}, p_{yz}) = p_{yx}, \tag{7}$$

or

$$\min(p_{xy}, p_{xz}) = p_{xy} \quad \text{and} \quad \min(p_{yx}, p_{yz}) = p_{yz}. \tag{8}$$

Equations (7) and (8) imply conditions 1 and 2, respectively. We now show that equation (7) implies condition 1; by symmetry, (8) implies condition 2. If (7) holds, then condition 3a gives

$$1 - p_{xz} - p_{yx} = p_{xy} - p_{xz} > p_{zx}p_{zy},$$

which is condition 1a.

We can rule out two cases for 3b as well. Of the remaining two, only the case where $\min(p_{yz}, p_{yx}) = p_{yx}$ and $\min(p_{zy}, p_{zx}) = p_{zy}$ is consistent with (7). Thus

$$1 - \min(p_{yz}, p_{yx}) - \min(p_{zy}, p_{zx}) = 1 - p_{yx} - p_{zy} = p_{yz} - p_{yx} > p_{xy}p_{xz},$$

which is condition 1b.

Finally, condition 3c gives

$$1 - \min(p_{zx}, p_{zy}) - \min(p_{xz}, p_{xy}) = 1 - p_{zy} - p_{xz} = p_{zx} - p_{zy} > p_{yx}p_{yz},$$

which is condition 1c. □

We do not know if Theorem 2.2 extends in some way to $|T| > 3$. One difficulty in trying to extend it is that the sufficiency of Suck's (2002) condition for an independent random utility representation is established only for $|T| = 3$.

3 Theorems for general universes

3.1 Extensions of binary systems to systems satisfying the MI

We will use the following theorem to show, constructively, that for any binary system p on a universe T, there is a system $P \in M_T(p)$; that is, a system on T extending p that satisfies the MI. However, the theorem is stronger than that: no matter how we choose $P \in M_T(p)$, and no matter how we extend p to a binary system p' on a larger universe T' containing T, we can extend p' to a system $P' \in M_{T'}(p')$ in such a way that P' is an extension of P to T'. The theorem reveals a kind of complementarity between the MI and axioms for binary choice probabilities.

Theorem 3.1 *Let T and $T' = T \cup \{y\}$, with $y \notin T$, be universes of choice objects. Let binary systems $p \in \delta_T$ and $p' \in \delta_{T'}$ be such that p' is an extension of p to T'. Let P be a system on T. Then if $P \in M_T(p)$, there exists a system $P' \in M_{T'}(p')$ that extends P to T'.*

Proof Suppose that $P \in M_T(p)$. For the given p', take any function $g_y\colon 2^T \to \mathbb{R}$ with the following properties:

1. For all $A \subseteq T$, $0 \leq g_y(A) \leq 1$.
2. $g_y(\emptyset) = 1$ and for all $x \in T$, $g_y(\{x\}) = p'_{yx}$.
3. For all $A, B \subseteq T$, $g_y(A)g_y(B) \leq g_y(A \cup B) \leq \min(g_y(A), g_y(B))$.

We can always take $g_y(A) = \prod_{x\in A} p_{yx}$ or $g_y(A) = \min_{x\in A} p_{yx}$, with the convention that $g_y(\emptyset) = 1$, both of which satisfy these properties.

We now extend P to a system P' on T' in the following way: for all $A \subseteq T$, let $P'_{A\cup\{y\}}(y) \equiv g_y(A)$ and for all $x \in A$, let $P'_{A\cup\{y\}}(x) \equiv (1 - g_y(A))P_A(x)$. Since $P \in \delta_T(p)$ and $P'_{\{x,y\}}(y) = p'_{yx}$ for all $x \in T$, $P' \in \delta_{T'}(p')$. We will call $g_y(\cdot)$ the *extension* function for the incremental element y.

Now let $A, B \subseteq T'$. If $y \notin A \cup B$, $P'_{A\cup B}(x) \geq P'_A(x)P'_B(x)$ by the assumption $P \in M_T(p)$ and the fact that P' is an extension of P. If $y \in A \cap B$, then

$$P'_{A\cup B}(y) = g_y(A \cup B - \{y\}) \geq g_y(A - \{y\})g_y(B - \{y\}) = P'_A(y)P'_B(y),$$

and for $x \in A \cap B$, $x \neq y$,

$$\begin{aligned} P'_{A\cup B}(x) &= (1 - g_y(A \cup B - \{y\}))P_{A\cup B-\{y\}}(x) \\ &\geq (1 - g_y(A \cup B - \{y\}))P_{A-\{y\}}(x)P_{B-\{y\}}(x) \\ &\geq (1 - g_y(A \cup B - \{y\}))(1 - g_y(A \cup B - \{y\}))P_{A-\{y\}}(x)P_{B-\{y\}}(x) \\ &\geq (1 - g_y(A - \{y\}))(1 - g_y(B - \{y\}))P_{A-\{y\}}(x)P_{B-\{y\}}(x) \\ &= P'_A(x)P'_B(x). \end{aligned}$$

The equalities come from the definition of the extension P'. The first inequality is by the MI for P on T. The second and third inequalities are from properties 1 and 3 of $g_y(\cdot)$. It may appear that the condition $g_y(A\cup B) \geq \min(g_y(A), g_y(B))$ could be relaxed to $1-g_y(A\cup B) \geq (1-g_y(A))(1-g_y(B)$, but that will not do for the remaining case.

Without loss of generality—we can switch A and B—the remaining case has $y \in A$ and $y \notin B$, where for all $x \in A \cap B$,

$$\begin{aligned} P'_{A\cup B}(x) &= (1 - g_y(A \cup B - \{y\}))P_{A\cup B-\{y\}}(x) \\ &\geq (1 - g_y(A \cup B - \{y\}))P_{A-\{y\}}(x)P_B(x) \\ &\geq (1 - g_y(A - \{y\}))P_{A-\{y\}}(x)P_B(x) \\ &= P'_A(x)P'_B(x). \square \end{aligned}$$

Corollary 3.2 *Let T be a universe of choice objects and let $p \in \delta_T$ be a binary system on T. Then there exists a system $P \in M_T(p)$.*

Proof We prove this by induction. Let p be a binary system on a universe T and let $n = |T|$. Let $\succ$ be any linear order on T and construct the sequence $T_1 \subset T_2 \subset \cdots \subset T_n = T$, where each T_i is the set of the i lowest $\succ$-ranked objects in T. We now show that there exists a sequence $P^{(1)}, \ldots, P^{(n)}$ of systems such that for $i = 1, \ldots, n$, $P^{(i)} \in M_{T_i}(p^{(i)})$, where $p^{(i)}$ is the restriction of p to T_i. There is exactly one system $P^{(1)}$ on T_1, and $P^{(1)} \in M_{T_1}(p_i)$ trivially. Theorem 3.1 provides the inductive step. The existence of $P^{(n)} \in M_{T_n}(p^{(n)}) = M_T(p)$ is the result we seek to prove.

We can interpret the final $P \in M_T(p)$ as giving, for each $x \in A \subseteq T$, the probability $P_A(x)$ of contestant x winning the following $|A|$-round sequential contest against the other contestants in A. The players in A are lined up in ascending $\succ$-order. In the first round, the first contestant is declared the provisional winner. The other players then take turns to try to supplant the current provisional winner to become the new provisional winner. On the turn of each contestant x, x becomes the provisional winner with probability $m_A(x) \equiv g_x(\{w \in A \colon x \succ w\})$. The (definitive) winner of the sequential contest is the provisional winner after the last contestant's turn. The probability that x is the winner is the probability that x becomes the provisional winner and that all higher $\succ$-ranked contestants in A fail to do so: thus

$$P_A(x) = m_A(x) \prod_{w \in A, x \succ w} (1 - m_A(w)).$$

For the special case

$$m_A(x) \equiv \prod_{w \in A,\, x \succ w} p_{xw},$$

we can think of $m_A(x)$ as the probability that x defeats *all* $\succ$-lower ranked opponents in A in statistically independent matches, where the probability of defeating any one such opponent w is p_{xw}. For the special case

$$m_A(x) \equiv \min_{w \in A,\, x \succ w} p_{xw},$$

x only has to defeat their most fearsome lower $\succ$-ranked opponent in A.

To clarify the connection between the contest and Theorem 3.1, we show directly that the system of probabilities induced by the contest satisfy the MI. The following properties of $m_A(x)$ are easy to check: $0 \leq m_A(x) \leq 1$ and for all $A, B \subseteq T$ and $x \in A \cap B$,

$$m_A(x) m_B(x) \leq m_{A \cup B}(x) \leq \min(m_A(x), m_B(x)).$$

Now for all $A, B \subseteq T$ and $x \in A \cap B$,

$$\begin{aligned}
P_A(x) P_B(x) &= m_A(x) m_B(x) \prod_{y \in A,\, y \succ x} (1 - m_A(y)) \prod_{y \in B,\, y \succ x} (1 - m_B(y)) \\
&\leq m_{A \cup B}(x) \prod_{y \in A,\, y \succ x} (1 - m_A(y)) \prod_{y \in B,\, y \succ x} (1 - m_B(y)) \\
&\leq m_{A \cup B}(x) \prod_{y \in A,\, y \succ x} (1 - m_{A \cup B}(y)) \prod_{y \in B,\, y \succ x} (1 - m_{A \cup B}(y)) \\
&\leq m_{A \cup B}(x) \prod_{y \in A \cup B,\, y \succ x} (1 - m_{A \cup B}(y)) = P_{A \cup B}(x). \square
\end{aligned}$$

One might conjecture that, for a given T and $p \in \delta_T$, every $P \in M_T(p)$ can be obtained using a sequential construction of the kind described in the proof of Corollary 3.2. The conjecture, more precisely, is the following. Suppose that T, $p \in \delta_T$ and $P \in M_T(p)$ are given. Then there is linear order $\succ$ on T and, for each $i = 2, \dots, n-1$, an extension function $g_{x_{i+1}} \colon 2^{T_i} \to \mathbb{R}$, where x_i is the i'th lowest $\succ$-ranked object in T and $T_i = \{x_1, \dots, x_i\}$, such that

1. $g_{x_{i+1}} \colon 2^{T_i} \to \mathbb{R}$ satifies conditions 1, 2 and 3 in the proof of Theorem 3.1, and
2. if we take the restriction of P to T_i and use the extension function $g_{x_{i+1}} \colon 2^{T_i} \to \mathbb{R}$ to extend it to P' as described in the proof of Theorem 3.1, then P' is the restriction of P to T_{i+1}.

However, the conjecture is false, even if we modify it to require $|T| = 3$. To provide counterexamples and illustrate the issues, consider the choice universe $T \equiv \{x, y, z\}$ and the binary system p on T defined by $p_{xy} = 0.4$, $p_{yz} = 0.7$ and $p_{xz} = 0.2$. Figure 3 and Table 1 help illustrate which $P \in M_T(p)$ can be constructed using Theorem 3.1 and which cannot. Each row of Table 1 provides a different ternary choice probability vector $P_T(\cdot)$ extending p to a system $P \in M_T(p)$; each row corresponds to a different combination of highest $\succ$-ranked object (last contestant) and extension function (giving its victory probability). Columns three through five specify $P_T(\cdot)$, both symbolically and numerically. Column one gives the name of the point in Figure 3 that represents $P_T(\cdot)$ in Barycentric coordinates. Column two gives two interchangeable linear orders $\succ$ that share the same highest ranked object and induce the same $P_T(\cdot)$ for a given extension function. Note that xyz, for example, represents the linear order $x \prec y \prec z$; this notation reflects the order of sequential construction (the order of contestants) and is the reverse of the notation that is customary when $\succ$ represents a preference. The vectors $P_T(\cdot)$ in the first three rows are constructed using the extension functions with $g_z(\{x, y\}) = p_{zx}p_{zy}$, $g_y(\{x, z\}) = p_{yx}p_{yz}$ and $g_x(\{y, z\}) = p_{xy}p_{xz}$, minimizing $P_T(z)$, $P_T(y)$ and $P_T(x)$, respectively, over the extension functions satisfying condition 3 in Theorem 3.1. The vectors $P_T(\cdot)$ in rows four through six are constructed using the extension functions with $g_z(\{x, y\}) = \min(p_{zx}, p_{zy})$, $g_y(\{x, z\}) = \min(p_{yx}, p_{yz})$ and $g_x(\{y, z\}) = \min(p_{xy}, p_{xz})$. These maximize the same probabilities.

Figure 3 represents, in Barycentric coordinates, the binary choice probabilities $p_{xy} = 0.4$, $p_{yz} = 0.7$ and $p_{xz} = 0.2$ and the six ternary choice probability vectors (points a, a', b, b', c, and c') of Table 1. The solid equilateral triangle in the interior of triangle xyz—points a, a', b, b', c, and c' all lie on its boundary—is the set of ternary probability vectors $P_T(\cdot)$ that extend p to a system satisfying the MI. We see, for example, that points a and a' minimize and maximize, respectively, the probability $P_T(z)$ among the extensions based on the orders $x \prec y \prec z$ and $y \prec x \prec z$ where object z is highest ranked and therefore introduced last; for these extensions, the constant ratio $P_T(x)/P_T(y) = p_{xy}/p_{yx}$ applies and the ternary probability $P_T(\cdot)$ always lies on the ray from z through a and a'. Points a and a' both lie on the boundary of the MI region, showing that the bounds on the function $g_z(\{x, y\})$ in condition 3 of Theorem 3.1 are sharp.

At the same time, however, the set of ternary probability vectors that are constructions described by Theorem 3.1, represented by the union of the line segments aa', bb' and cc', extend p to a small subset of $M_T(p)$. Even the convex hull of these points, representing mixtures of those choice distributions, corresponds to a strict subset of $M_T(p)$.

The importance of Theorem 3.1 goes beyond establishing the existence of extensions of binary systems to systems satisfying the MI and doing so constructively. It also shows that the sequential construction of systems satisfying the MI as new choice objects are introduced never leads to a dead end. The system $P \in M_T(p)$ in the preamble of the theorem is arbitrary. It is not necessarily one that can be constructed only using repeated applications of the theorem, and we can choose it without regard to the binary choice probabilities p'_{xy}, $x \in T$, extending p to p' on $T' \equiv T \cup \{y\}$.

Point	$\succ$	$P_T(x)$	$P_T(y)$	$P_T(z)$
a	xyz/yxz	$p_{xy}(1-p_{zx}p_{zy})=0.304$	$p_{yx}(1-p_{zx}p_{zy})=0.456$	$p_{zx}p_{zy}=0.24$
b	xzy/zxy	$p_{xz}(1-p_{yx}p_{yz})=0.116$	$p_{yx}p_{yz}=0.42$	$p_{zx}(1-p_{yx}p_{yz})=0.464$
c	yzx/zyx	$p_{xy}p_{xz}=0.08$	$p_{yz}(1-p_{xy}p_{xz})=0.644$	$p_{zy}(1-p_{xy}p_{xz})=0.276$
a'	xyz/yxz	$p_{xy}(1-p_{zx}\wedge p_{zy})=0.28$	$p_{yx}(1-p_{zx}\wedge p_{zy})=0.42$	$p_{zx}\wedge p_{zy}=0.3$
b'	xzy/zxy	$p_{xz}(1-p_{yx}\wedge p_{yz})=0.08$	$p_{yx}\wedge p_{yz}=0.6$	$p_{zx}(1-p_{yx}\wedge p_{yz})=0.32$
c'	yzx/zyx	$p_{xy}\wedge p_{xz}=0.2$	$p_{yz}(1-p_{xy}\wedge p_{xz})=0.56$	$p_{zy}(1-p_{xy}\wedge p_{xz})=0.24$

Table 1: Ternary choice probability vectors $P_T(\cdot)$ completing the binary system p defined by $p_{xy}=0.4$, $p_{yz}=0.7$ and $p_{xz}=0.2$. Each row is a different extension of p to a system satisfying the MI. We use the notation $\wedge$ for the binary minimum operator.

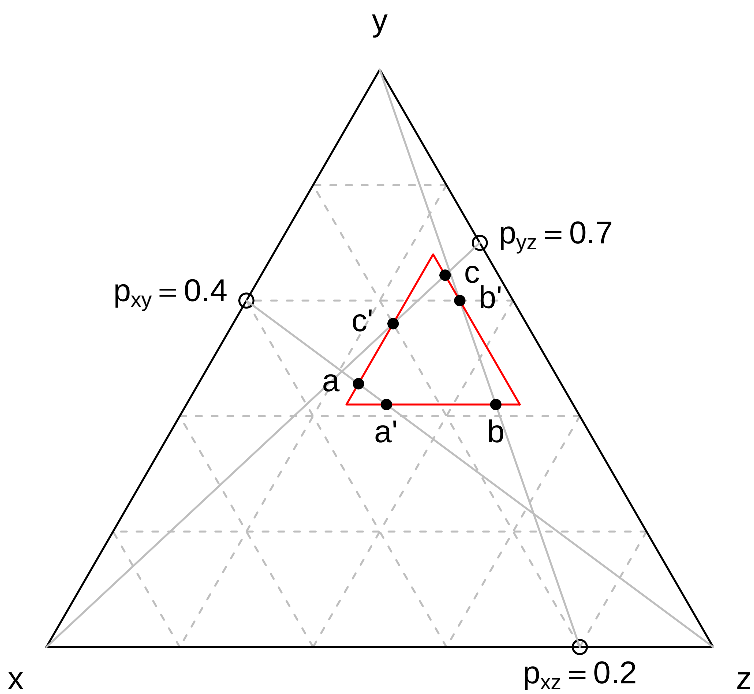

Figure 3: Ternary choice probability vectors extending the binary system p defined by $p_{xy}=0.4$, $p_{yz}=0.7$ and $p_{xz}=0.2$ to a system satisfying the MI. Each of the points a, a', b, b', c and c' corresponds to a row of Figure 3 and extends p to a system satisfying the MI. The equilateral triangle on whose boundary the six points fall is the set of ternary probability vectors extending p to a system satisfying the MI.

This flexibility of sequential extensions is a strong and important property of the MI. To illustrate this, we will show that regularity does not have the analogous property. Recall Sprumont's (2022) result that any binary system satisfying the triangle inequality can be extended to a system satisfying regularity. For regular extensions to have the same kind of flexibility that extensions satisfying the MI have, the following conjecture would have to be true. Let T and $T' = T \cup \{w\}$, with $w \notin T$, be universes of choice objects. Then for all $p \in \delta_T$ satisfying the triangle inequality on T, all $p' \in \delta_{T'}$ extending p to T' and satisfying the triangle inequality on T', and all $P \in R_T(p)$, there exists a $P' \in R_{T'}(p')$ extending P to T'.

However, the conjecture is false, as the following counterexample shows. Let $T \equiv \{x, y, z\}$ and $T' \equiv \{w, x, y, z\}$, define p on T by setting $p_{yx} = p_{yz} = p_{zx} = \frac{1}{2}$, and extend p to p' on T' by setting $p'_{wx} = p'_{yw} = p'_{zw} = \frac{1}{4}$. It is easy to check that p', and therefore p, satisfy the triangle inequality, but note that the particular inequalities $p'_{yw} + p'_{wx} \geq p_{yx}$ and $p'_{zw} + p'_{wx} \geq p_{zx}$ hold with equality, which implies that there is only one ternary probability vector consistent with both p' and regularity on each of the choice sets $\{w, x, y\}$ and $\{w, x, z\}$, respectively; these are

$$(P'_{\{w,x,y\}}(w), P'_{\{w,x,y\}}(x), P'_{\{w,x,y\}}(y)) = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}) \tag{9}$$

and

$$(P'_{\{w,x,z\}}(w), P'_{\{w,x,z\}}(x), P'_{\{w,x,z\}}(z)) = (\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}). \tag{10}$$

This means that any extension of p' to a regular system P' on T' must satisfy (9), (10), and therefore

$$P'_{\{w,x,y,z\}}(w) \leq \tfrac{1}{4}, \quad P'_{\{w,x,y,z\}}(y) \leq \tfrac{1}{4}, \quad P'_{\{w,x,y,z\}}(z) \leq \tfrac{1}{4}. \tag{11}$$

Now consider the extension of p to a regular system P. One such extension is obtained by setting $(P_{\{x,y,z\}}(x), P_{\{x,y,z\}}(y), P_{\{x,y,z\}}(z)) = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$. A regular extension of this system to T' would require $P'_{\{w,x,y,z\}}(x) \leq \frac{1}{5}$, which is inconsistent with (11) and the adding-up constraint. Therefore there is no such extension. Even though choosing $(P_{\{x,y,z\}}(x), P_{\{x,y,z\}}(y), P_{\{x,y,z\}}(z)) = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5})$ gives a regular system on T, it is a dead end.

Of course, there must be *some* choice of $P_{\{x,y,z\}}(\cdot)$ such that we can extend P to a regular system P' on T' that is also an extension of p'. One such choice is $(P_{\{x,y,z\}}(x), P_{\{x,y,z\}}(y), P_{\{x,y,z\}}(z)) = (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$ and one of the extensions of this to T' is defined by (9), (10),

$$(P'_{\{w,y,z\}}(w), P'_{\{w,y,z\}}(y), P'_{\{w,y,z\}}(z)) = (\tfrac{3}{5}, \tfrac{1}{5}, \tfrac{1}{5}),$$

and

$$(P'_{\{w,x,y,z\}}(w), P'_{\{w,x,y,z\}}(x), P'_{\{w,x,y,z\}}(y), P'_{\{w,x,y,z\}}(z)) = (\tfrac{1}{5}, \tfrac{2}{5}, \tfrac{1}{5}, \tfrac{1}{5}).$$

3.2 Non-convexity of the MI region

We have seen that for any given binary system p on $T = \{x, y, z\}$, the cross sections $M_T(p)$ and $R_T(p)$ are convex. It is also easy to see that for all T, R_T is convex: Δ_T is convex and R_T is the intersection of Δ_T and a collection of half-spaces. It might therefore be tempting to conjecture that M is also convex. However, it is not, for $|T| \geq 3$.

Theorem 3.3 *For all universes T such that $|T| \geq 3$, M_T is not convex.*

Proof First take $T = \{x, y, z\}$. The following two systems satisfy the MI:

$$p_{xy} = p_{yz} = p_{zx} = 1,\ P_{\{x,y,z\}}(x) = 1,\ P_{\{x,y,z\}}(y) = 0,\ P_{\{x,y,z\}}(z) = 0,$$

and

$$p_{xy} = p_{yz} = p_{zx} = 0,\ P_{\{x,y,z\}}(x) = 1,\ P_{\{x,y,z\}}(y) = 0,\ P_{\{x,y,z\}}(z) = 0.$$

The convex combination of the two, with equal weights, is the system

$$p_{xy} = p_{yz} = p_{zx} = \tfrac{1}{2}, P_{\{x,y,z\}}(x) = 1, P_{\{x,y,z\}}(y) = 0, P_{\{x,y,z\}}(z) = 0,$$

which fails to satisfy the MI since $P_{x,y,z}(y) = 0 < p_{yx}p_{yz} = \frac{1}{4}$. Now take any finite T' such that $T \subseteq T'$. By repeated application of Theorem 3.1 we can extend the first two systems on T, above, to T' in such a way that the MI is satisfied. The convex combination of the two extensions, with equal weights, will always violate the MI. □

The non-convexity of M_T has at least two important implications. The first is theoretical. It is possible for every individual in a population to choose according to a system of choice probabilities satisfying the MI and for the aggregate system of choice probabilities (a convex combination of individuals' systems of choice probabilities) not to satisfy the MI. A second implication is computational. In numerical optimization or simulation problems, it is advantageous for the domain of optimization or simulation to be convex. If one is trying to compute supremum estimators or to simulate posterior distributions in the space M_T of systems satisfying MI, one cannot, unfortunately, rely on a convex domain.

3.3 Redundant multiplicative inequality conditions

We have already pointed out a kind of complementarity between regularity and the MI. The following theorem shows that if regularity is imposed, some of the instances of the multiplicative inequality become redundant. The instances that remain more concisely describe the additional theoretical content that the MI contributes over regularity alone.

Theorem 3.4 *Suppose P is a random choice structure on master set T. Then P satisfies regularity and the multiplicative inequality if and only if the following two conditions hold:*

1. *For all $A \subseteq T$, $x \in A$ and $y \in A^c$, $P_A(x) \geq P_{A\cup\{y\}}(x)$.*
2. *For all $A \subseteq T$, $x \in A$ and $B \subseteq A^c$, $P_{A\cup B}(x) \geq P_A(x)P_{B\cup\{x\}}(x)$.*

Proof We begin with the necessity of conditions 1 and 2. If P satisfies regularity and the multiplicative inequality, then condition 1 follows directly from the definition of regularity and condition 2 follows directly from the definition of the multiplicative inequality. We now establish the sufficiency of the conditions. Condition 1 is obviously sufficient for regularity, by repeated application. Now suppose P satisfies conditions 1 and 2. Then for all $A, B \subseteq T$ and all $x \in A \cap B$,

$$\begin{aligned} P_{A\cup B}(x) &\geq P_A(x)P_{(B\cap A^c)\cup\{x\}}(x) \\ &\geq P_A(x) \cdot P_B(x). \end{aligned}$$

The first inequality follows from condition 2. The second is obtained by repeated application of condition 1. □

The theorem has some theoretical interest, but also has some practical implications. In some numerical applications, it is necessary to check whether or not a given system satisfies a particular axiom or condition. For example, Bayesian and frequentist statistical tests of a given condition require exploration of the space of systems, and checking that condition at many points in that space. The theorem tells us that if we want to check whether both the MI and regularity hold, it suffices to check a much smaller number of instances of the MI.

For a universe T with n elements, the number of instances of the multiplicative inequalities is $n2^{2(n-1)}$: there are n possible objects x, and for each x, there are 2^{n-1} choices for each of the menus A and B in the definition of MI. For the purposes of verifying the MI for a particular system, we can deduct from this total the number of cases where $A \subset B$ or $B \subset A$, cases for which the inequality always holds. Also, the interchangeability of A and B makes half of the remaining cases redundant. The theorem tells us that if regularity holds, we can further restrict the search to A and B whose intersection is $\{x\}$.

4 Random preference models inconsistent with MI

Here we expand on the hypothetical random preference example discussed in Section 1.2, put forward by Corbin and Marley (1974), where options are ranked $x \succ y \succ z$ or $z \succ y \succ x$, each with probability $\frac{1}{2}$, and the induced choice probabilities violate the MI.

We begin by defining a random preference model on $T = \{x, y, z\}$ as the vector $(\pi_{xyz}, \pi_{xzy}, \pi_{yxz}, \pi_{yzx}, \pi_{zxy}, \pi_{zyx})$ of preference probabilities, where for distinct $a, b, c \in T$, π_{abc} is the probability of preference $a \succ b \succ c$. The induced choice probabilities are $p_{xy} = \pi_{xyz} + \pi_{xzy} + \pi_{zxy}$, $p_{yz} = \pi_{yzx} + \pi_{yxz} + \pi_{xyz}$, $p_{zx} = \pi_{zxy} + \pi_{zyx} + \pi_{yzx}$, and $(P_T(x), P_T(y), P_T(z)) = (\pi_{xyz} + \pi_{xzy}, \pi_{yxz} + \pi_{yzx}, \pi_{zxy} + \pi_{zyx})$.

We now write the MI condition for the system P induced by a random preference model directly in terms of the preference probabilities. The MI implies $P_T(x) \geq p_{xy}p_{xz}$, which can be expressed, after some manipulation, as

$$(\pi_{xyz} + \pi_{xzy})(\pi_{yzx} + \pi_{zyx}) \geq \pi_{zxy}\pi_{yxz}.$$

Similarly, $P_T(y) \geq p_{yx}p_{yz}$ and $P_T(z) \geq p_{zx}p_{zy}$ can be expressed as

$$(\pi_{yxz} + \pi_{yzx})(\pi_{xzy} + \pi_{zxy}) \geq \pi_{xyz}\pi_{zyx}, \tag{12}$$

$$(\pi_{zxy} + \pi_{zyx})(\pi_{xyz} + \pi_{yxz}) \geq \pi_{xzy}\pi_{yzx}.$$

At first glance, these inequalities seem quite reasonable. Each of the right hand sides of these inequalities is the product of probabilities of two rankings, each of which is the reverse of the other.[1] It might seem likely that if either of these probabilities is high, then the other is likely to be small.

Corbin and Marley's (1974) example gives $(\pi_{yxz} + \pi_{yzx})(\pi_{xzy} + \pi_{zxy}) = 0$ and $\pi_{xyz}\pi_{zyx} = \frac{1}{4}$, which violates equation (12). Here we suggest that such examples may be widespread, but first we define single-peaked preferences, introduced by Black (1948). Suppose we have a universe $T = \{w_1, \ldots, w_n\}$, with $n \geq 3$, endowed by some objective strict ordering R of objects. For example, each w_i is a level of government spending. Without loss of generality, we assign indices so that $w_1 R w_2 R \cdots R w_n$. Let $\succ$ be some preference on T and suppose, without loss of generality, that w_i is the most preferred element according to $\succ$. Then $\succ$ is *single-peaked* relative to R if $w_i \succ w_{i+1} \succ \cdots \succ w_n$ and $w_i \succ w_{i-1} \succ \cdots \succ w_1$. Any single peaked preference over levels of government spending is understandable, even if it does not agree with our own. Other preferences, which rank an intermediate level below two levels on either side, seem bizarre.

Now consider a random preference on T that puts positive probability on every single-peaked preference and zero probability on all others. The random preference might apply to an individual

[1] This makes the two rankings maximally distant from each other by Kendall's τ metric and the Ulam distance, although not by the Cayley and Hamming distances.

or a population of individuals. The system of probabilities induced by the random preference violates the MI. To see this, take the restriction of the random preference to $\{x, y, z\} \subseteq T$ and suppose, without loss of generality, that $xRyRz$. Then the second factor of the LHS of (12) must be zero (the middle element can never be the least preferred of a single-peaked preference) and the RHS must be positive.

5 Conclusions

The usual way to reconcile deterministic choice models with stochastic data is to introduce random preferences or error terms describing deviations between the predictions of deterministic models and observed choices. These approaches are convenient, but not usually justified by any theory. If we want to rise to Luce's challenge and build choice models that are supported by stochastic foundations and that generalize existing deterministic models, one natural approach is to introduce one or more axioms constraining binary choice probabilities and one or more axioms constraining choice probabilities across sets of different sizes. As far as binary choice probabilities go, there are many stochastic analogues of transitivity to choose from. One difficulty is that the relationship between binary choice and multiple choice in stochastic environments is much more complicated. We have surveyed some of the existing axioms and conditions constraining multiple choice probabilities: regularity, the MI, the constant ratio rule, simple scalability and random utility.

The MI condition, despite being simple, plausible, testable and necessary for the EBA model, has received relatively little attention, and one of the goals of this paper has been to shed more light on it. Theorem 3.1 guarantees that binary systems can be extended to systems satisfying MI, and the extensions can be constructed sequentially with no danger of getting stuck in dead ends. The theorem reveals an appealing compatibility between MI and any axioms for binary choice probabilities one might propose. The MI is a natural condition to structure the relationships between binary and mulitiple choice probabilities because, unlike other well known conditions, it does not constrain binary choice probabilities.

We have also shed light on the relationship between regularity and the MI. We have illustrated the complementarity between the two conditions graphically. We have shown two intriguing results for the special case where $|T| = 3$, an important case because many context effects are framed as conditions relating choice probabilities across different subsets of a tripleton set. The first of these results, Theorem 2.1, states that if there is a regular extension of a binary system on T then there is a regular extension satisfying the MI. While not all regular systems satisfy the MI, imposing the MI does not preclude the regular extension of any binary system when such an extension is possible without it. This goes some way to establishing compatibility between regularity and the MI. It is an open question whether this result generalizes to arbitrary T, as in the following conjecture, extending both Theorem 2.1 and Sprumont's (2022) result that any binary system satisfying the triangle inequality can be extended to a regular system: if a binary system p on an arbitrary universe T satisfies the triangle inequality then there is a system P extending p that satisfies both regularity and the MI. A constructive proof of this conjecture based on Theorem 3.1, perhaps imposing constraints on $\succ$ or the $g_y(\cdot)$, appears to have some promise: the monotonicity of the extension $P'_A(x)$ of Theorem 3.1, as A varies for fixed $x \neq y$, is automatically satisfied, and the resemblance of equation (3) to condition 3 of the theorem is intriguing. On the other hand, the monotonicity of $P'_A(y)$ is in the wrong direction, and any constructive proof of the conjecture would also be a constructive proof of Sprumont's (2022) weaker result; comments in his paper and the counterexample near the end of Section 3.1 of the current paper suggest that even this may be difficult. The second result for the special case $|T| = 3$, Theorem 2.2, provides a geometric, and somewhat counter-intuitive, condition that is equivalent to Suck's (2002) necessary and sufficient condition for an independent random utility representation. Unfortunately, the sufficiency of Suck's (2002) condition has not been established for $|T| > 3$, which makes it difficult to try to extend our result to general T. We have shown that while the set of choice probabilities satisfying regularity is convex, the set satisfying the MI is not. Another result shows that with regularity maintained, many instances of the multiplicative inequality are redundant. These last two results are of interest to anyone doing empirical testing of the MI and related constraints, or model-free estimation of choice probabilities subject to these constraints, where it is important to check whether a given system satisfies regularity and the MI.

The paper supports the advice that the MI should not stand alone in any stochastic model. Theorem 3.1 implies that the MI fails to rule out any systems of binary choice probabilities, however unrealistic. Furthermore, the results and observations in Section 2.2 suggest that a large fraction of the volume of M_T is taken up by implausible systems of choice probabilities, having such features as stochastically intransitive cycles of binary choice probabilities and large departures from regularity. The paper also gives reasons to consider regularity and suitable binary choice axioms as candidate additional conditions.

We have also provided some conditions on random preferences where violations of the MI are bound to occur. The result suggests that in environments where we expect single peaked preferences to be more plausible than other preferences, we should not be surprised to find violations of the MI.

McCausland is planning future work testing the MI (conjointly with regularity and conjointly with various conditions on binary choice probabilities) using a wide variety of small choice universes, where for each universe we collect choice frequency data for all doubleton and larger subsets of the universe and thereby expose all implications of the MI to possible falsification.

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A Axioms and conditions

The system of choice probabilities P on the universe T satisfies

weak stochastic transitivity if for all distinct $x, y, z \in T$,

$$p_{xy} \geq \tfrac{1}{2} \text{ and } p_{yz} \geq \tfrac{1}{2} \Rightarrow p_{xz} \geq \tfrac{1}{2};$$

moderate stochastic transitivity if for all distinct $x, y, z \in T$,

$$p_{xy} \geq \tfrac{1}{2} \text{ and } p_{yz} \geq \tfrac{1}{2} \Rightarrow p_{xz} \geq \min[p_{xy}, p_{yz}];$$

strong stochastic transitivity if for distinct $x, y, z \in T$,

$$p_{xy} \geq \tfrac{1}{2} \text{ and } p_{yz} \geq \tfrac{1}{2} \Rightarrow p_{xz} \geq \max[p_{xy}, p_{yz}];$$

the triangle inequality if for distinct $x, y, z \in T$,

$$p_{xy} + p_{yz} \geq p_{xz};$$

constant ratio rule if for all $x, y \in A$,

$$\frac{p_{xy}}{p_{yx}} = \frac{P_A(x)}{P_A(y)},$$

whenever the denominators do not vanish;

order independence if for all $A, B \subseteq T$, $x, y \in A - B$ and $z \in B$

$$P_A(x) \geq P_A(y) \Leftrightarrow P_{B \cup \{x\}}(z) \leq P_{B \cup \{y\}}(z),$$

provided the choice probabilities on the two sides of either inequality are not both 0 or 1;

the *Block-Marschak* conditions if for all $x \in A \subseteq T$,

$$\sum_{B:\, A \subseteq B \subseteq T} (-1)^{|B \setminus A|} P_B(x) \geq 0; \tag{13}$$

References

Black, D. (1948). 'On the Rationale of Group Decision-making', *Journal of Political Economy*, 56: 23—34.

Block, H. D., and Marschak, J. (1960). 'Random Orderings and Stochastic Theories of Responses', in I. Olkin, S. G. Ghurye, W. Hoeffding, W. G. Madow, and H. B. Mann (eds.), *Contributions to Probability and Statistics: Essays in Honor of Harold Hotelling*, pp. 97–132. Stanford University Press, Stanford, CA.

Colonius, H. (1983). 'A characterization of stochastic independence by association, with an application to random utility theory', *Journal of Mathematical Psychology*, 27(1): 103–105.

Corbin, R., and Marley, A. A. J. (1974). 'Random utility models with equality: an apparent, but not actual, generalization of random utility models', *Journal of Mathematical Psychology*, 11: 274–293.

Dasgupta, I., and Pattanaik, P. (2007). ''Regular' choice and the weak axiom of stochastic revealed preference', *Economic Theory*, 1(35-50).

Davis-Stober, C. P., Marley, A. A. J., McCausland, W. J., and Turner, B. M. (2023). 'An illustrated guide to context effects', *Journal of Mathematical Psychology*, 115: In press.

Falmagne, J. C. (1978). 'A Representation Theorem for Finite Random Scale Systems', *Journal of Mathematical Psychology*, 18: 52–72.

Fishburn, P. C. (1999). 'Stochastic Utility', in S. Barberà, P. J. Hammond, and C. Seidl (eds.), *Handbook of Utility Theory: Volume 1, Principles*, chap. 7, pp. 273–319. Kluwer Academic Publishers, Dordrecht, The Netherlands.

Huber, J., Payne, J. W., and Puto, C. (1982). 'Adding Asymmetrically Dominated Alternatives: Violations of Regularity and the Similarity Hypothesis', *Journal of Consumer Research*, 9: 90–98.

Krantz, D. H. (1964). 'The scaling of small and large color differences.', Ph.D. thesis, University of Pennsylvania.

Luce, R. D. (1959). *Individual Choice Behavior: A Theoretical Analysis.* John Wiley & Sons, New York, NY.

——— (1995). 'Four tensions concerning mathematical modeling in psychology', *Annual Review of Psychology*, 46: 1–26.

——— (1997). 'Several unresolved conceptual problems of mathematical psychology', *Journal of Mathematical Psychology*, 41: 79–87.

Luce, R. D., and Suppes, P. (1965). 'Preference, Utility, and Subjective Probability', in R. D. Luce, R. R. Bush, and E. Galanter (eds.), *Handbook of Mathematical Psychology*, vol. 3, chap. 19, pp. 249–410. John Wiley & Sons, New York, NY.

McCausland, W. J., Davis-Stober, C. P., Marley, A. A. J., Park, S., and Brown, N. (2020). 'Testing the Random Utility Hypothesis Directly', *The Economic Journal*, 130: 183–207.

McCausland, W. J., and Marley, A. A. J. (2013). 'Prior Distributions for Random Choice Structures', *Journal of Mathematical Psychology*, 57: 78–93.

——— (2014). 'Bayesian Inference and model comparison for random choice structures', *Journal of Mathematical Psychology*, 62-63: 33–46.

McFadden, D. (1981). 'Econometric models of probabilistic choice', in C. Manski and D. McFadden (eds.), *Structural Analysis of Discrete Data with Econometric Applications*, pp. 198—272. MIT Press, Cambridge, MA.

Rieskamp, J., Busemeyer, J. R., and Mellers, B. A. (2006). 'Extending the bounds of rationality: Evidence and theories of preferential choice', *Journal of Economic Literature*, 44: 631–661.

Sattath, S., and Tversky, A. (1976). 'Unite and Conquer: A Multiplicative Inequality for Choice Probabilities', *Econometrica*, 44: 79–89.

Simonson, I. (1989). 'Choice Based on Reasons: The Case of Attraction and Compromise Effects', *Journal of Consumer Research*, 16.

Sprumont, Y. (2022). 'Regular random choice and the triangle inequalities', *Journal of Mathematical Psychology*, 110: 1–8.

Suck, R. (2002). 'Independent random utility representations', *Mathematical Social Sicences*, 43: 371–389.

Tversky, A. (1972a). 'Choice by Elimination', *Journal of Mathematical Psychology*, 9: 341–367.

Tversky, A. (1972b). 'Elimination by Aspects: A Theory of Choice', *Psychological Review*, 79(4): 281–299.

Tversky, A., and Simonson, I. (1993). 'Context-Dependent Preferences', *Management Science*, 39: 1179–1189.